TORIC DEGENERATIONS, FANO SCHEMES AND COMPUTATIONS IN TROPICAL GEOMETRY

by

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Contents

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To my family and my friends a very big thank you for being my rock in these years.

Declarations

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself, except where stated otherwise, and it has not been submitted in any previous application for any degree or award. Chapter 4 has been carried out by the author only ([Lam]), the work of Chapter 3 is an adapted version of [BLMM17] and it is joint with Lara Bossinger, Kalina Mincheva and Fatemeh Mohammadi, Chapter 5 is a collaboration with Carlos Améndola, Kathlén Kohn, Diane Maclagan, Benjamin Smith, Jeff Sommars, Paolo Tripoli, and Magdalena Zajaczkowska ([AKL+17]).

Abstract

Tropical geometry is a developing area of mathematics in between algebraic geometry, combinatorics and polyhedral geometry. The main objects of study are tropical varieties which can be seen as polyhedral and combinatorial shadows of classical algebraic varieties. These simpler geometric objects keep important geometric information. They can be used to tackle problems in algebraic geometry but also to develop a purely tropical theory which has connection to different areas of research.

In Chapter 3 we study toric degenerations of the full flag varieties $\mathcal{F}\ell_4$ and $\mathcal{F}\ell_5$ using their tropicalization and we compare these with toric degenerations coming from representation theory techniques, namely degenerations associated to *string polytopes* and the *Feigin-Fourier-Littelmann-Vinberg polytope*.

The classical Fano scheme $\mathrm{F}_d(X)$ of a projective variety $X \subset \mathbb{P}^n$ parametrises d-dimensional linear spaces contained in X. In Chapter 4 we study tropical versions of the Fano scheme and their relations with the classical $F_d(X)$. One version is the tropicalization $\operatorname{trop} \mathrm{F}_d(X)$ while the second $\mathrm{F}_d(\operatorname{trop} X)$ has a completely tropical construction. An interesting problem is to understand when these two versions coincide. We address this problem for some specific varieties such as linear spaces, toric varieties and hypersurfaces.

Computations with tropical varieties are at the basis of tropical geometry. In Chapter 5 we present a Macaulay2 package `Tropical.m2` that we developed in order to provide a user friendly tool to do these computations in Macaulay2.

Chapter 1

Introduction

Tropical geometry is a new area of mathematics at the intersection of algebraic geometry, combinatorics and polyhedral geometry. At the heart of tropical geometry there are tropical varieties that can be seen as a combinatorial and polyhedral shadow of the classical algebraic varieties. The operation of tropicalization allows to pass from classical to tropical geometry. Although the obtained tropicalized varieties have a simpler structure they still carry meaningful information of the original variety. This is the reason why tropical geometry has been used to study problems in algebraic geometry ([Mik05],[Gro11],[Gro10],[RSS14],[CDPR12],[Mar08]). However, the introduction of this new geometry also gives rise to many interesting problems that are purely tropical ([MR18],[GG16],[Cha17]). In some cases these have also unexpected connections with different areas of research such as optimization theory, mathematical biology and economics ([SS09],[BK18],[Kri17]).

There are many approaches to tropical geometry and different equivalent ways of obtaining the tropicalization of algebraic varieties ([Mik04],[BPR16]). In this thesis we refer to the approach in [MS15] where the tropicalization is obtained from the polynomials defining the classical variety(see Chapter 2).

The three main themes of this thesis are toric degenerations of flag varieties (Chapter 3), Fano schemes and their tropical versions (Chapter 4) and computations with tropical varieties using computer algebra software (Chapter 5).

Toric degenerations of flag varieties and Schubert varieties have been studied intensively over the last two decades (see [FFL16a] for an overview on this topic). These are obtained using different techniques and there are many studies that compare the different results. In Chapter 3 we study toric degenerations of the full flag varieties $\mathcal{F}\ell_n$ for $n = 4, 5$ using their tropicalization and we compare these with the toric degenerations coming from representation theory techniques, namely degenerations associated to *string polytopes* and the *Feigin-Fourier-Littelmann-Vinberg polytope (FFLV polytope).* The flag variety $\mathcal{F}\ell_n$ is the variety parametrising full flags

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = \mathbb{C}^n$$

of vector subspaces of $\mathbb{C}^n$ where $\dim_{\mathbb{C}}(V_i) = i$. A 1-parameter toric degeneration of $\mathcal{F}\ell_n$ is a flat family $\varphi : \mathcal{F} \to \mathbb{A}^1$, where the fibre over zero (also called the *special* fibre) is a toric variety and all other fibres are isomorphic to $\mathcal{F}\ell_n$. We produce toric degenerations of $\mathcal{F}\ell_n$ as Gröbner degenerations coming from the initial ideals associated to the maximal cones of $\operatorname{trop} \mathcal{F}\ell_n$.

The following are the main results of Chapter 3. We will call a maximal cone C of $\operatorname{trop} X$ *prime* if $\operatorname{in}_C(I) := \operatorname{in}_{\mathbf{w}}(I)$ is prime, with $\mathbf{w}$ a vector in the relative interior of C.

Theorem 1. *The tropical variety* $\operatorname{trop} \mathcal{F}\ell_4 \subset \mathbb{R}^{14}/\mathbb{R}^3$ *is a 6-dimensional fan with* 78 *maximal cones. From prime cones we obtain four non-isomorphic toric degenerations. After applying Procedure 1 we obtain at least two additional non-isomorphic toric degenerations from non-prime cones.*

Theorem 2. *The tropical variety* $\operatorname{trop} \mathcal{F}\ell_5 \subset \mathbb{R}^{30}/\mathbb{R}^4$ *is a 10-dimensional fan with* 69780 *maximal cones. From prime cones we obtain 180 non-isomorphic toric degenerations.*

Theorem 3. *For* $\mathcal{F}\ell_4$ *there is at least one new toric degeneration arising from prime cones of* $\operatorname{trop} \mathcal{F}\ell_4$ *in comparison to those obtained from string polytopes and the FFLV polytope.*

For $\mathcal{F}\ell_5$ *there are at least 168 new toric degenerations arising from prime cones of* $\operatorname{trop} \mathcal{F}\ell_5$ *in comparison to those obtained from string polytopes and the FFLV polytope.*

In the forth chapter we study the classical Fano scheme of a projective variety $X \subset \mathbb{P}^n$. This is the fine moduli space parametrising linear spaces contained in X. It is denoted by $\mathrm{F}_d(X)$ and it is a subvariety of the Grassmannian $\mathbb{G}(d, n)$ that parametrises d-dimensional linear spaces in $\mathbb{P}^n$. Fano schemes have been intensively studied because they encode many geometric properties of the variety X. In Chapter 4 we study tropical versions of the Fano scheme. We investigate the structure of this tropical object and the relations with the classical $F_d(X)$. The first way of obtaining a tropical version of $\mathrm{F}_d(X)$ is to consider its tropicalization inside $\operatorname{trop}\mathrm{Gr}(d,n)$. The points of $\operatorname{trop}\mathrm{F}_d(X)$ are in correspondence with the tropicalization of the classical linear spaces contained in X. However it is not true in general that a tropicalized linear space that lies in $\operatorname{trop}X$ is the tropicalization of a classical linear space in X. This leads us to define the *tropical Fano scheme* $\mathrm{F}_d(\operatorname{trop}X)$ to be the set of tropicalized linear spaces Γ of dimension d contained in $\operatorname{trop}X$. In particular we will focus on $d = 1$. We take the first steps in studying the properties of this object that can be used to study the classical Fano scheme and the geometric properties of the tropical variety $\operatorname{trop}X$. The theorem below allows us to consider the dimension of $\mathrm{F}_1(\operatorname{trop}X)$ which gives a bound for the dimension of $F_1(X)$.

Theorem 4. *Let X be a projective variety in $\mathbb{P}^n$. Then the tropical Fano scheme $\mathrm{F}_1(\operatorname{trop}X)$ is a polyhedral complex whose support is contained in $\operatorname{trop}\mathrm{Gr}(1,n)$.*

The two tropical versions of the Fano scheme $\operatorname{trop}\mathrm{F}_1(X)$ and $\mathrm{F}_1(\operatorname{trop}X)$ come from two different constructions. The first is strictly linked to the algebraic variety and to its classical Fano scheme while the other only depends on the tropical variety $\operatorname{trop}X$. However we immediately observe that

$$\operatorname{trop}\mathrm{F}_1(X) \subset \mathrm{F}_1(\operatorname{trop}X) \tag{1.1}$$

and a natural question arises:

Question 5. *For which varieties X do we have $\operatorname{trop}\mathrm{F}_1(X) = \mathrm{F}_1(\operatorname{trop}X)$?*

An answer to this question gives a deep insight on the connection between X and $\operatorname{trop}X$. We start by looking at some particular algebraic varieties: linear subspaces of $\mathbb{P}^n$ and toric varieties.

Theorem 6.

1. *If L is a generic 2−dimensional plane in $\mathbb{P}^5$ then $\operatorname{trop}\mathrm{F}_1(L) \subsetneq \mathrm{F}_1(\operatorname{trop}L)$.*
2. *If X is a toric variety in $\mathbb{P}^n$ then $\mathrm{F}_1(\operatorname{trop}X) = \operatorname{trop}\mathrm{F}_1(X)$.*

In Chapter 5 we present a `Macaulay2` package `Tropical` that we developed in order to provide a user friendly tool to do tropical computations in Macaulay2. The main tool to tropicalize an algebraic variety defined by an ideal is `Gfan` by Jensen. In Macaulay2 the package `gfanInterface2` provides an interface to `Gfan`. The tropical computations can be performed with it but they require a good knowledge of functions and conventions of `Gfan`. The `Tropical` package instead does not require any previous knowledge of the processes in the background necessary for the operations. It includes different strategies for the same function depending on the input and calls functions from `Gfan`, via `gfanInterface2`, and `Polymake`, as appropriate.

Chapter 2

Preliminary background on tropical geometry

This chapter contains preliminary material on tropical geometry. In the first two sections, §2.1 and §2.2, we introduce the basic objects of polyhedral geometry and valuations on fields. These are the building blocks of tropical varieties. In section 2.3 we define the tropicalization of subvarieties of the torus and then in section 2.5 we show how to extend this to subvarieties of toric varieties. We describe the tropicalization of the Grassmannian in section 2.4. The content of this chapter is not original and it is mostly based on the approach of [MS15].

2.1 Polyhedral geometry

The three basic objects of polyhedral geometry are *polyhedra, polytopes* and *cones.*

Note 2.1.1. *We always assume $\mathbb{R}^n$ has the Euclidean topology.*

Notation 2.1.2. We denote by $\mathbf{e}_1, \dots, \mathbf{e}_n$ the standard basis vectors of $\mathbb{R}^n$.

Definition 2.1.3. A *polyhedron* P in $\mathbb{R}^n$ is the intersection of finitely many half-spaces, that is $P = \{\mathbf{x} \in \mathbb{R}^n : A \cdot \mathbf{x} \leq \mathbf{b}\}$, where A is an $m \times n$ matrix, $\mathbf{b} \in \mathbb{R}^m$ and $A \cdot \mathbf{x} \leq \mathbf{b}$ is the system of inequalities $\{a_{i,1}x_1 + \dots + a_{i,n}x_n \leq b_i\}_{i=1}^m$. A *polytope* is a bounded polyhedron.

Definition 2.1.4. Let Γ be a subgroup of $(\mathbb{R}, +)$. A $\Gamma-$*rational polyhedron* is $P = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}\}$ where A is an $m \times n$ matrix with entries in $\mathbb{Q}$ and $\mathbf{b} \in \Gamma^m$.

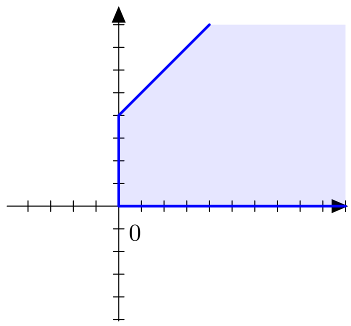

Figure 2.1: An unbounded polyhedron in $\mathbb{R}^2$.

We say that an inequality $c_1x_1 + \ldots + c_nx_n \le d$ is *valid* for P if for every $\mathbf{p} = (p_1, \ldots, p_n) \in P$ we have $c_1p_1 + \ldots + c_np_n \le d$ and there exists at least a point $q \in P$ such that $c_1q_1 + \ldots + c_nq_n = d$. From now on we will always assume that the inequalities defining a polyhedron P are all valid.

Let $U = \{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ be a finite subset of $\mathbb{R}^n$.

Definition 2.1.5. The *convex hull* of the points in U is denoted by $\operatorname{conv}(U)$ and is the set

$$\operatorname{conv}(U) = \left\{ \sum_{i=1}^{r} \lambda_i \mathbf{u}_i : 0 \le \lambda_i \le 1, \sum_{i=1}^{r} \lambda_i = 1 \right\}.$$

A *cone* C is a set $C = \operatorname{pos}(U)$ where

$$\operatorname{pos}(U) = \{ \sum_{i=1}^{r} \lambda_i \mathbf{u}_i \in \mathbb{R}^n : \lambda_i \ge 0 \textit{ for all } i \}.$$

If $U = \{\mathbf{v}\}$ we denote $\operatorname{pos}(U)$ by $\operatorname{pos}(\mathbf{v})$ and we call it a *ray*.

The following theorem relates these three objects (see [Zie95, Theorem 1.1, Theorem 1.2, Theorem 1.3]).

Theorem 2.1.6. *A set $P \subset \mathbb{R}^n$ is a polyhedron if and only if $P = \operatorname{conv} U + \operatorname{pos} V$ where U and V are finite sets of points in $\mathbb{R}^n$.*
Moreover if $U = \varnothing$ then $P = \{\boldsymbol{x} \in \mathbb{R}^n : A \cdot \boldsymbol{x} \le 0\}$ with A an $m \times n$ matrix.

From Theorem 2.1.6 we deduce that convex hulls and cones are special types of polyhedra.

Example 2.1.7. The polytope P in Figure 2.2 is the convex hull of

$$U = \{(0,3), (-4,0), (2,1), (2,-3), (-3,-2)\}.$$

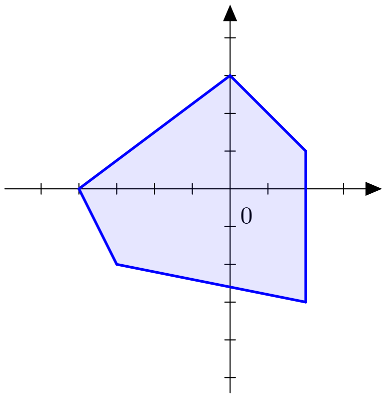

Figure 2.2: A polytope in $\mathbb{R}^2$.

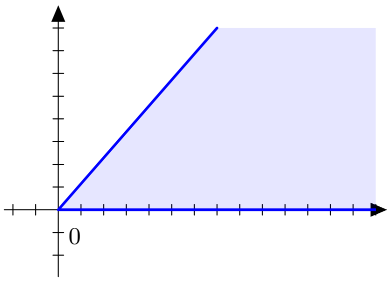

Figure 2.3: A cone in $\mathbb{R}^2$.

It can also be described as the set $\{\mathbf{x} \in \mathbb{R}^n : A \cdot \mathbf{x} \leq \mathbf{b}\}$ where

$$A = \begin{pmatrix} -3 & 4 \\ 1 & 1 \\ -5 & -1 \\ -1 & -2 \\ 1 & 0 \end{pmatrix} \quad \text{and } b = (12, 3, 13, 8, 1).$$

Definition 2.1.8. The *lineality space* of a polyhedron P is the largest subspace V in $\mathbb{R}^n$ with the property that if $x \in P$ and $\mathbf{v} \in V$ then $x + \mathbf{v} \in P$.
The *affine span* of a polyhedron P is the smallest affine subspace that contains P. This is parallel to the *linear space parallel* to P denoted by $\text{span}(P)$.
The *relative interior* of P is denoted by P°. It is the interior of P inside its affine span.
The *dimension* of P is the dimension of $\text{span}(P)$.

Definition 2.1.9. Let P be a polyhedron and $\mathbf{w} \in (\mathbb{R}^n)^\vee = \mathrm{Hom}(\mathbb{R}^n, \mathbb{R})$. If $\mathbf{w}\cdot\mathbf{x} \leq w_0$ is satisfied by all points in P then a *face* of P is the set

$$\mathrm{face}_{\mathbf{w}}(P) = P \cap \{\mathbf{x} \in \mathbb{R}^n : \mathbf{w} \cdot \mathbf{x} = w_0\}.$$

For $\mathbf{0} \cdot \mathbf{x} \leq 0$ we get P as a face while for $\mathbf{0} \cdot \mathbf{x} \leq 1$ we have the empty set. Any face distinct from P is a *proper* face of P. The zero dimensional faces are called vertices and the one dimensional faces are called edges. The *facets* are the maximal dimensional proper faces of P.

If $\mathbf{w} \cdot \mathbf{x} \leq w_0$ is a valid inequality for P then $\mathrm{face}_{\mathbf{w}}(P)$ is non-empty.

Remark 2.1.10. Let $P = \{\mathbf{x} \in \mathbb{R}^n : A \cdot \mathbf{x} \leq \mathbf{b}\}$ be a full dimensional polytope. From Theorem 2.1.6 we deduce that $P^\circ = \{\mathbf{x} \in \mathbb{R}^n : A \cdot \mathbf{x} < \mathbf{b}\}$. If P is not full dimensional then $P \subset H$ with H a hyperplane of $\mathbb{R}^n$ whose dimension is $\dim P$. The matrices A and $\mathbf{b}$ can be partitioned in two submatrices A', A'' and $\mathbf{b}', \mathbf{b}''$ such that $\{\mathbf{x} \in \mathbb{R}^n : A' \cdot \mathbf{x} = \mathbf{b}'\} = H$ and then $P^\circ = \{\mathbf{x} \in \mathbb{R}^n : A' \cdot \mathbf{x} = \mathbf{b}', A'' \cdot \mathbf{x} < \mathbf{b}''\}$.

Example 2.1.11. Let P be the polyhedron defined by the following matrices A and $\mathbf{b}$

$$A = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ -1 \end{pmatrix}.$$

This is a polytope contained in $H = \{(x, y) \in \mathbb{R}^2 : y = 1\}$. Let A' be the matrix $\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$ and $\mathbf{b}'$ be the vector $(1, -1)$, then $A'' = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$, $\mathbf{b}'' = (1, 2)$ and $P^\circ = \{(x, y) \in \mathbb{R}^2 : x \leq 1, -x \leq 2, y = 1\}$.

Definition 2.1.12. A *polyhedral complex* Σ is a collection of polyhedral cones that satisfies the following properties:

1. the empty polyhedron is in Σ;

2. every face of a polyhedron $P \in \Sigma$ is in Σ;

3. the intersection of any two polyhedra in Σ is a face of each.

If all the polyhedra in Σ are cones then Σ is a *fan*.

We will denote by $|\Sigma|$ the *support* of Σ that is the set $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \in \ P \in \Sigma\}$. The polyhedra in Σ are called the *cells* of the polyhedral complex. Cells that are not

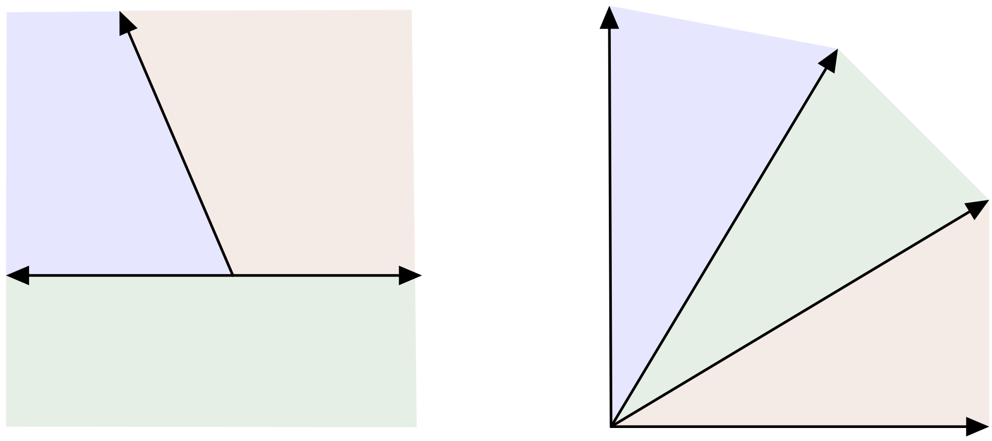

Figure 2.4: An example of a non polyhedral complex (on the left) and of a fan (on the right).

faces of any larger cell are called *facets*. The lineality space of a polyhedral complex Σ is the intersection of all the lineality spaces of the polyhedra in Σ.
The dimension of a polyhedral complex Σ is defined to be

$$\dim P = \max_{P \in \Sigma}\{d : \dim P = d\}.$$

A polyhedral complex where all the facets have the same dimension is called *pure*. In this case we say that the polyhedral complex is *connected through codimension 1* if for every two distinct facets P, P' there is a chain $P = P_1, P_2, \ldots, P_s = P'$, where P_i and P_{i+1} are cells and share a common facet for all $i = 1, \ldots, s-1$.
A *balanced weighted* polyhedral complex is a polyhedral complex such that each one of its maximal cells is labelled with a positive integer, called the multiplicity, and there is a balancing condition that holds (see [MS15, Definition 3.3.1]).
In the case of a 1-dimensional fan Σ we can define the balancing condition as follows:

Definition 2.1.13. Let Σ be a 1-dimensional fan with rays $\mathrm{pos}(\mathbf{v}_1), \ldots, \mathrm{pos}(\mathbf{v}_s)$, where $\mathbf{v}_i$ is a primitive vector for $i = 1, .., s$, and multiplicities $m_1, \ldots, m_s$. We say that the fan is balanced if $\sum_{i=1}^{s} m_i \cdot \mathbf{v}_i = 0$.

Let P be a polytope. The collection Σ of its faces is an example of a polyhedral complex.

An important example of fan is the *normal fan* to a given polytope.

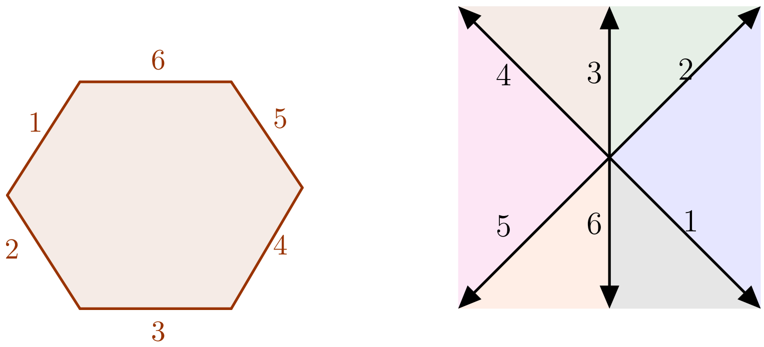

Figure 2.5: An example of a polytope (on the left) and its inner normal fan (on the right). The rays of the fan are labelled as their normal edge.

Definition 2.1.14. Let $P \subset \mathbb{R}^n$ be a polyhedron. The *(inner) normal fan* of P is the polyhedral fan $\mathcal{N}_P$ consisting of the cones $\mathcal{N}_P(F) = \overline{\{\mathbf{w} \in (\mathbb{R}^n)^\vee : \text{face}_{\mathbf{w}}(P) = F\}}$ with F a face of P. The cone $\mathcal{N}_P(F)$ (*resp.* the face F) is called the *normal* or *dual* cone to F (*resp.* normal or dual face to $\mathcal{N}_P(F)$).

Remark 2.1.15. The *outer* normal fan is the negative of the inner normal fan.

The following construction is often used in tropical geometry (see section 2.3).

Definition 2.1.16. Let P be the polytope in $\mathbb{R}^n$ with vertices $\mathbf{u}_1, \dots, \mathbf{u}_r$ and let $\mathbf{w}$ be a weight vector $\mathbf{w} = (w_1, \dots, w_r) \in \mathbb{R}^r$. Consider the polytope

$$P_{\mathbf{w}} = \text{conv}\{(\mathbf{u}_i, w_i) : 1 \le i \le r\}$$

and its lower faces (those with an inner normal vector $c \in \mathbb{R}^{n+1}$ with positive last coordinate). The *regular subdivision* $\Delta_{\mathbf{w}}$ of P induced by $\mathbf{w}$ is the polyhedral complex given by the projection of the lower faces of $P_{\mathbf{w}}$ onto P.

Definition 2.1.17. The *dual complex* to the regular subdivision $\Delta_{\mathbf{w}}$ of a polytope P is the polyhedral complex whose faces are $\tilde{\pi}(\mathcal{N}(F))$ where $\mathcal{N}(F) = \mathcal{N}_{P_{\mathbf{w}}}(F)$, F is a lower face of $P_{\mathbf{w}}$ and $\tilde{\pi}$ is the restriction of the projection to the vectors $(\mathbf{v}, 1) \in \mathcal{N}(F)$. Each $\mathbf{v}$ is dual to a face of $P_{\mathbf{w}}$.

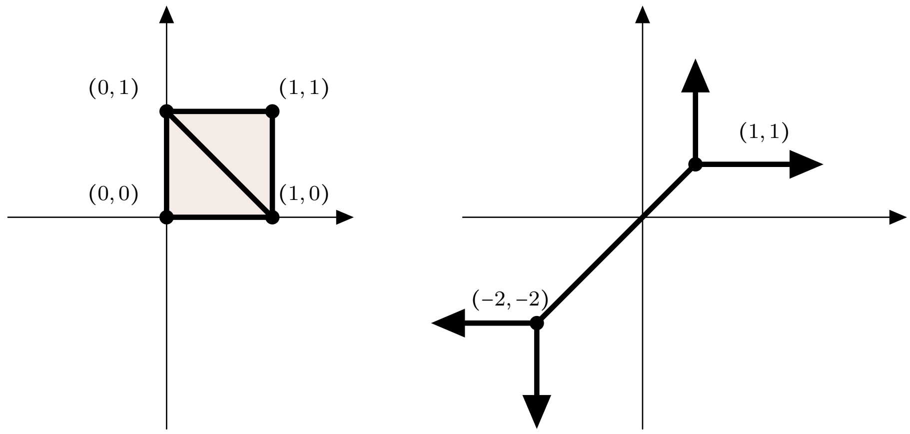

Figure 2.6: The regular subdivision (on the right) $P_{\mathbf{w}}$ and its dual complex as in Example 2.1.19

Remark 2.1.18. We have that $\mathbf{c}$ is in $\tilde{\pi}(\mathcal{N}(F))$ if and only if there exists $c_0 > 0$ such that $(\mathbf{c},1)\cdot(\mathbf{u}_i,w_i) = c_0$ for $i \in F$ and $(\mathbf{c},1)\cdot(\mathbf{u}_i,w_i) > c_0$ for $i \notin F$. This is equivalent to $(-\mathbf{c},c_0)\cdot(\mathbf{u}_i,1) = w_i$ for $i \in F$ and $(-\mathbf{c},c_0)\cdot(\mathbf{u}_i,1) < w_i$ for $i \notin F$. Hence we can solve these systems of equations $(-\mathbf{c},c_0)\cdot(\mathbf{u}_i,1) = w_i$ for $i \in F$ for all lower faces of F to get the dual complex to the regular subdivision of a polytope P.

Example 2.1.19. Consider the set of points $U = \{(0,0),(1,0),(0,1),(1,1)\}$ in $\mathbb{R}^2$ and $\mathbf{w} = (4,2,2,3)$ a weight vector. In order to compute the regular subdivision $\Delta_{\mathbf{w}}$ of $\mathrm{conv}(U)$ we consider $P_{\mathbf{w}} = \mathrm{conv}(\{(0,0,4),(1,0,2),(0,1,2),(1,1,3)\})$. The lower facets are $\mathrm{conv}(\{(0,0,4),(1,0,2),(0,1,2)\})$ and $\mathrm{conv}(\{(1,0,2),(0,1,2),(1,1,3)\})$. We project these down and we obtain the regular subdivision of $P_{\mathbf{w}}$ shown in Figure 2.6. Using Remark 2.1.18 it is possible to compute the dual complex shown in Figure 2.6.

The last concept we introduce is the *stable intersection* of two pure weighted balanced polyhedral fans.

Let N be the lattice $\mathbb{Z}^n$ of $\mathbb{R}^n$ and σ be a Γ−polyhedron. We can associate to σ a lattice N_σ given by the sublattice of N generated by the lattice points contained in the linear space parallel to σ. Moreover for any N' sublattice of N with the same rank we can define the index $[N:N']$, that is the order of the quotient N/N'.

Definition 2.1.20. Let Σ_1 and Σ_2 be two polyhedral complexes in $\mathbb{R}^n$ and let $\mathbf{w} \in |\Sigma_1| \cap |\Sigma_2|$. The point $\mathbf{w}$ lies in the relative interior of a unique cell σ_i in Σ_i for $i = 1, 2$. We say that the complexes Σ_1 and Σ_2 *meet transversally* at $\mathbf{w}$ if the affine span of σ_i is $\mathbf{w} + L_i$ and $L_1 + L_2 = \mathbb{R}^n$.

Given two pure weighted balanced polyhedral complexes Γ_1 and Γ_2 such that their polyhedral complex structures intersect transversally at every point in $\Gamma_1 \cap \Gamma_2$, we can give to the intersection the structure of weighted balanced polyhedral complexes. In fact we can associate to each maximal cone $\sigma_1 \cap \sigma_2$ multiplicity

$$\mathrm{mult}_{\Sigma_1}(\sigma_1)\,\mathrm{mult}_{\Sigma_2}(\sigma_2)[N : N_{\sigma_1} + N_{\sigma_2}]. \tag{2.1}$$

Example 2.1.21. Let Σ_1 be the fan in $\mathbb{R}^2$ given by the rays $\mathrm{pos}(\mathbf{e}_1), \mathrm{pos}(\mathbf{e}_2)$ and $\mathrm{pos}(-\mathbf{e}_1 - \mathbf{e}_2)$ with origin in $(0, 0)$ and multiplicity 1 on each of the rays and let Σ_2 be the fan with the same rays and multiplicity but origin in $p = (2, -2)$ (see Figure 2.7). They intersect only at the point $(2, 0)$ and the intersection is transverse since the two linear space parallel to the two rays sum up to $\mathbb{R}^2$. We can compute the multiplicity using the formula (2.1):

$$\mathrm{mult}_{\Sigma_1 \cap \Sigma_2}(p) = \mathrm{mult}_{\Sigma_1}(\sigma_1)\,\mathrm{mult}_{\Sigma_2}(\sigma_2)[N : N_{\sigma_1} + N_{\sigma_2}] = 1 \cdot 1 \cdot [\mathbb{Z}^2 : N_{\sigma_1} + N_{\sigma_2}] = 1$$

where $[\mathbb{Z}^2 : N_{\sigma_1} + N_{\sigma_2}] = |\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}| = 1$.

We have to introduce stable intersection in order to have a good notion of intersection for polyhedral complexes. In fact the definition of multiplicity in (2.1) can not be applied when the two polyhedra do not meet transversally.

Example 2.1.22. Consider Σ_1 as in the previous example and Σ_2 be the fan whose cones are $\mathrm{pos}(\mathbf{e}_1 + \mathbf{e}_2), \mathrm{pos}(-\mathbf{e}_1, -\mathbf{e}_2)$ and the multiplicity is 1 on each ray. These two fans do not intersect transversally along the ray $\mathrm{pos}(-\mathbf{e}_1 - \mathbf{e}_2)$.

Definition 2.1.23. Let Σ be a polyhedral complex in $\mathbb{R}^n$ and σ be a cell of Σ. The *star* of σ in Σ is a fan in $\mathbb{R}^n$, denoted by $\mathrm{star}_\Sigma(\sigma)$. Its cones are indexed by those cells $\tau \in \Sigma$ that contain σ as a face. In particular a cone of $\mathrm{star}_\Sigma(\sigma)$ is the set $\{\lambda(\mathbf{x} - \mathbf{y}) : \lambda > 0; \mathbf{x} \in \tau, \mathbf{y} \in \sigma\}$.

Definition 2.1.24. Let Σ_1 and Σ_2 be pure weighted balanced polyhedral complexes

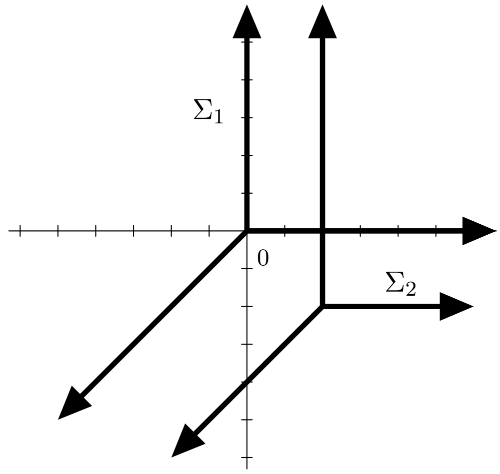

Figure 2.7: The two fans Σ_1 and Σ_2 of Example 2.1.21.

in $\mathbb{R}^n$. The *stable intersection* $\Sigma_1 \cap_{st} \Sigma_2$ is the polyhedral complex

$$\Sigma_1 \cap_{st} \Sigma_2 = \bigcup_{\substack{\sigma_1 \in \Sigma_1, \sigma_2 \in \Sigma_2 \\ \dim(\sigma_1 + \sigma_2) = n}} \sigma_1 \cap \sigma_2. \tag{2.2}$$

The multiplicity of a top-dimensional cell $\sigma_1 \cap \sigma_2$ in $\Sigma_1 \cap_{st} \Sigma_2$ is

$$\mathrm{mult}_{\Sigma_1 \cap_{st} \Sigma_2}(\sigma_1 \cap \sigma_2) = \sum_{\tau_1, \tau_2} \mathrm{mult}_{\Sigma_1}(\tau_1)\, \mathrm{mult}_{\Sigma_2}(\tau_2)[N : N_{\tau_1} + N_{\tau_2}] \tag{2.3}$$

where the sum is over all $\tau_1 \in \mathrm{star}_{\Sigma_1}(\sigma_1 \cup \sigma_2)$ and $\tau_2 \in \mathrm{star}_{\Sigma_2}(\sigma_1 \cup \sigma_2)$ with $\tau_1 \cap (\mathbf{v} + \tau_2) \neq \varnothing$ for some fixed generic $\mathbf{v}$.
This is independent of the choice of $\mathbf{v}$ (see [MS15] section 3.6).

Note that in the case where the two polyhedral complexes meet transversally we have that $\Sigma_1 \cap \Sigma_2 = \Sigma_1 \cap_{st} \Sigma_2$ with the multiplicities defined above.

Example 2.1.25. The stable intersection of Σ_1 and Σ_2 in Example 2.1.22 is given by the point $(0, 0)$ with multiplicity 1.

2.2 Valuations

Let $\mathbb{K}$ be a field.

Definition 2.2.1. A *valuation* val is a map $\mathrm{val} : \mathbb{K} \to \mathbb{R} \cup \{\infty\}$ such that for any $x, y \in \mathbb{K}$ the following properties are satisfied:

1. $\mathrm{val}(x) = \infty$ if and only if $x = 0$;
2. $\mathrm{val}(xy) = \mathrm{val}(x) + \mathrm{val}(y)$;
3. $\mathrm{val}(x + y) \geq \min\{\mathrm{val}(x), \mathrm{val}(y)\}$.

Note that it is possible to determine the valuation of $\mathrm{val}(x + y)$ whenever $\mathrm{val}(x) \neq \mathrm{val}(y)$, in fact $\mathrm{val}(x+y) = \min\{\mathrm{val}(x), \mathrm{val}(y)\}$. The image of val is denoted by Γ_{val} and it is an additive subgroup of $\mathbb{R}$. The set $\mathfrak{R} = \{x \in \mathbb{K} : \mathrm{val}(x) \geq 0\}$ is a local ring with maximal ideal $\mathfrak{m} = \{x \in \mathbb{K} : \mathrm{val}(x) > 0\}$. The field $k = \mathfrak{R}/\mathfrak{m}$ is called the *residue field* associated to the valuation. The image of val is the *value group* Γ_{val}.

For any field $\mathbb{K}$ the map $\mathrm{val} : \mathbb{K} \to \mathbb{R} \cup \{\infty\}$ that sends all nonzero x to 0 is a valuation. It is called the *trivial* valuation.

An important example of a field with a nontrivial valuation is the field of *Puiseux series* with coefficients in $\mathbb{C}$ denoted by $\mathbb{C}\{\{t\}\}$. The elements of $\mathbb{C}\{\{t\}\}$ are formal power series

$$c(t) = c_0 t^{a_0} + c_1 t^{a_1} + \dots$$

where $c_i \in \mathbb{C}$ are non zero for all i and the $(a_i)_{i \in \mathbb{N}}$ is a strictly increasing sequence of rational numbers with a common denominator. This is an algebraically closed field and the valuation val is the map that sends $c(t) \neq 0$ to a_0. For example $c(t) = \sum_{i=2}^{\infty} 3it^i$ has valuation 2 while $c(t) = \sum_{i=0}^{\infty} 3it^{i-1}$ has valuation 0. The field $\mathbb{C}$ is contained in the Puiseux series and $(\mathrm{val})|_{\mathbb{C}}$ is the trivial valuation.
This field is one of the most used in tropical geometry since it contains $\mathbb{C}$ and it has nontrivial valuation which is necessary for fundamental results on tropical varieties.

Remark 2.2.2. If we substitute $\mathbb{C}$ with any other field $\mathbb{K}$ we obtain the Puiseux series with coefficients in $\mathbb{K}$. If $\mathbb{K}$ has characteristic zero and it is algebraically closed then the same holds for $\mathbb{K}\{\{t\}\}$ but if the characteristic of $\mathbb{K}$ is not zero then $\mathbb{K}\{\{t\}\}$ is not algebraically closed. Given a divisible subgroup $G \subset \mathbb{R}$ we can extend the construction of Puiseux series. The Mal'cev-Neumann ring $\mathbb{K}((G))$ of generalised power series is the field whose elements are of the form $\sum_{\alpha \in A} c_\alpha t^\alpha$ with $c_\alpha \in \mathbb{K}$ and

$A \subset G$ well ordered. This is also algebraically closed ([Pas85, Theorem 13.2.11], [Poo93, Corollary 4]. In Chapter 4 we will use the field $\mathbb{C}((\mathbb{R}))$.

Lemma 2.2.3. *Let $\mathbb{K}$ be an algebraically closed field with nontrivial valuation. Then Γ_{val} is a dense subgroup of $\mathbb{R}$.*

Another important example of a field with a nontrivial valuation val is $\overline{\mathbb{Q}(t)}$. This is a subfield of $\mathbb{C}\{\{t\}\}$ and the valuation is the restriction of the valuation on $\mathbb{C}\{\{t\}\}$. From Lemma 2.2.3 we deduce that Γ_{val} is dense in $\mathbb{R}$. Hence if we are only interested in $\overline{\Gamma}_{\text{val}}$ it is sufficient to consider $\overline{\mathbb{Q}(t)}$ instead of $\mathbb{C}\{\{t\}\}$ that is also more suitable for computations.

2.3 Tropicalization I: subvarieties of the torus T^n

Let $\mathbb{K}$ be a field with valuation val and k be the residue field associated to val.

Let $f = \sum_{\mathbf{u} \in U \subset \mathbb{Z}^n} c_{\mathbf{u}} \mathbf{x}^{\mathbf{u}}$ be a Laurent polynomial in $\mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.

Definition 2.3.1. The *tropicalization* of f is the real-valued function $\text{trop}(f) : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\text{trop}(f)(\mathbf{w}) = \min_{\mathbf{u} \in U \subset \mathbb{Z}^n} \{\text{val}(c_{\mathbf{u}}) + \sum_{i=1}^{n} u_i w_i\} = \min_{\mathbf{u} \in U \subset \mathbb{Z}^n} (\text{val}(c_{\mathbf{u}}) + \mathbf{u} \cdot \mathbf{w}).$$

We now define the tropicalization of subvarieties of the torus $(\mathbb{K}^*)^n$.

Let $V(f) = \{\mathbf{y} \in T^n : f(\mathbf{y}) = 0\}$ be the hypersurface in T^n defined by f.

Definition 2.3.2. The *tropical hypersurface* $\text{trop}\, V(f)$ is the set

$$\{\mathbf{w} \in \mathbb{R}^n : \text{the minimum in } \text{trop}(f)(\mathbf{w}) \text{ is attained at least two times}\}.$$

In other words, it is the locus of points at which the function $\text{trop}(f)$ is no longer a linear function.

In general, let I be an ideal in $\mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ and $X = V(I)$ be the subvariety of T^n associated to I.

Definition 2.3.3. The *tropicalization* of X is denoted by $\text{trop}\, X$. It is the set

$$\text{trop}(X) = \bigcap_{f \in I} \text{trop}(V(f)). \tag{2.4}$$

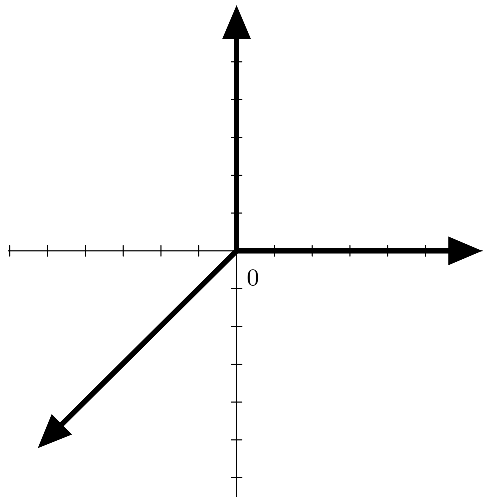

Figure 2.8: The tropicalization $\operatorname{trop} V(f)$ as in Example 2.3.4.

Example 2.3.4. Let $f = tx + ty + t$ be the polynomial in $\mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}]$. The tropicalization of $V(f)$ is

$$\begin{aligned}\operatorname{trop} V(f) = \{(u_1, u_2) : u_1 + 1 = u_2 + 1 \leq 1\} \cup \\ \{(u_1, u_2) : u_1 + 1 = 1 \leq u_2 + 1\} \cup \\ \{(u_1, u_2) : u_2 + 1 = 1 \leq u_1 + 1\}.\end{aligned}$$

We call this the *standard tropical line* in $\mathbb{R}^2$ (see Figure 2.8).

The same definitions can be given for ideals and polynomials in $\mathbb{K}[x_0, \dots, x_n]$. Hence we can define the tropicalization of subvarieties of $\mathbb{P}^n$ and $\mathbb{A}^n$. In this case we observe that if I or f contain a monomial then the tropicalization of the associated projective or affine variety would be empty.

Note that if f or I are homogeneous in $\mathbb{K}[x_0, \dots, x_n]$ then $\operatorname{trop} V(I)$ will contain the 1-dimensional linear space $\mathbb{R}\mathbf{1} = \operatorname{span}(1, 1, \dots, 1)$. Hence we consider $\operatorname{trop} V(I)$ as a polyhedral complex in $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1} \cong \mathbb{R}^n$, with isomorphism

$$\begin{aligned}\phi : \mathbb{R}^{n+1}/\mathbb{R}\mathbf{1} &\longrightarrow \mathbb{R}^n \\ (x_0, \dots, x_n) &\mapsto (x_1 - x_0, \dots, x_n - x_0).\end{aligned} \tag{2.5}$$

In the following whenever we identify $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ with $\mathbb{R}^n$ using this isomorphism we have that $\phi(\mathbf{e}_i) = \mathbf{e}_{i-1}$ for $i = 2, \dots, n+1$ and $\phi(\mathbf{e}_1) = \mathbf{e}_0 := -\sum_{i=1}^{n+1} \mathbf{e}_i$.

Remark 2.3.5. Let $V(I)$ be a subvariety of $\mathbb{P}^n$. If $V(I)$ is reducible then the tropicalization will be nonempty only for those components which intersect the torus $T^n = (\mathbb{K}^*)^{n+1}/\mathbb{K}^*$. Hence $\operatorname{trop} V(I) = \operatorname{trop}(V(I) \cap T^n)$. In section 2.5 we show how to tropicalize all components of a subvariety of $\mathbb{P}^n$ introducing the *extended tropicalization*.

Example 2.3.6. Let $g = tx + ty + tz$ be the polynomial in $\mathbb{C}\{\{t\}\}[x, y, z]$. The tropicalization of $V(f)$ is

$$\operatorname{trop} V(f) = \{(u_1, u_2, u_3) : u_1 = u_2 \le u_3\} \cup \\ \{(u_1, u_2, u_3) : u_1 = u_3 \le u_2\} \cup \{(u_1, u_2, u_3) : u_2 = u_3 \le u_1\}.$$

If we quotient by $\mathbb{R}\mathbf{1}$ we obtain the same set as in Example 2.3.4.

It is important to note that in general the intersection in (2.4) can not be substituted with the intersection over polynomials $f_1, \dots, f_s$ that generate the ideal I. The intersection $\operatorname{trop} V(f_1) \cap \dots \cap \operatorname{trop} V(f_n)$ is called the *tropical prevariety* associated to I.

Definition 2.3.7. A finite generating set B_I of an ideal $I \subset \mathbb{K}[x_1^{\pm1}, \dots, x_n^{\pm1}]$ is called a *tropical basis* for I if $\operatorname{trop}(V(I)) = \bigcap_{f \in B_I} \operatorname{trop}(V(f))$.

A proof of the existence of a tropical basis for any ideal of $I \subset \mathbb{K}[x_1^{\pm1}, \dots, x_n^{\pm1}]$ can be found in Theorem 2.6.6 in [MS15].

The tropicalization can also be defined using *initial forms* and *initial ideals*. In the following we assume that $\operatorname{val} : \mathbb{K} \to \Gamma_{\operatorname{val}}$ splits, that is there exists $\psi : \Gamma_{\operatorname{val}} \to \mathbb{K}$ such that $\operatorname{val}(\psi(w)) = w$ for $w \in \Gamma_{\operatorname{val}}$. If $\mathbb{K}$ is algebraically closed then this is always the case ([MS15, Lemma 2.1.15]). If $w \in \Gamma_{\operatorname{val}}$ we denote $\psi(w)$ by t^w.

Let $f = \sum_{\mathbf{u} \in U \subset \mathbb{Z}^n} c_{\mathbf{u}} \mathbf{x}^{\mathbf{u}}$ be a Laurent polynomial in $\mathbb{K}[x_1^{\pm}, \dots, x_n^{\pm}]$ and let $\mathbf{w}$ be a vector in $\mathbb{R}^n$.

Definition 2.3.8. The *initial form* $\operatorname{in}_{\mathbf{w}}(f)$ is the polynomial in $k[x_1^{\pm1}, \dots, x_n^{\pm1}]$ defined by

$$\operatorname{in}_{\mathbf{w}}(f) = \sum_{\substack{\mathbf{u} \in U \\ \operatorname{val}(c_{\mathbf{u}}) + \mathbf{u} \cdot \mathbf{w} = \operatorname{trop}(f)(\mathbf{w})}} \overline{t^{-\operatorname{val}(c_{\mathbf{u}})} c_{\mathbf{u}}} \mathbf{x}^{\mathbf{u}}$$

where $\overline{t^{-\operatorname{val}(c_{\mathbf{u}})} c_{\mathbf{u}}}$ is the image of $t^{-\operatorname{val}(c_{\mathbf{u}})} c_{\mathbf{u}}$ under the projection map $\mathfrak{R} \to \mathfrak{R}/\mathfrak{m} \cong k$. Let I be an ideal contained in $\mathbb{K}[x_1^{\pm}, \dots, x_n^{\pm}]$. The *initial ideal* $\operatorname{in}_{\mathbf{w}}(I)$ is the ideal $(\operatorname{in}_{\mathbf{w}}(f) : f \in I) \subset \mathbb{K}[x_1^{\pm1}, \dots, x_n^{\pm1}]$.

Theorem 2.3.9. *[Fundamental theorem of tropical algebraic geometry] Let* $\mathbb{K}$ *be an algebraically closed field with nontrivial valuation. Let* I *be an ideal of* $\mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ *and* $X = V(I) \subset T^n$*. Then the following three sets are equal:*

1. $\operatorname{trop} X$;
2. $\{\mathbf{w} \in \mathbb{R}^n : \operatorname{in}_{\mathbf{w}}(I) \neq (1)\}$;
3. $\operatorname{val}(X) = \overline{\{(\operatorname{val}(y_1), \dots, \operatorname{val}(y_n)) : (y_1, \dots, y_n) \in X\}}$.

Moreover, if X is irreducible and $\mathbf{w} \in \Gamma_{val}^n \cap \operatorname{trop}(X)$, then $\{\mathbf{y} \in X : \operatorname{val}(\mathbf{y}) = \mathbf{w}\}$ is Zariski dense in X.

Remark 2.3.10. If $\mathbb{K}$ is not algebraically closed then we can consider its algebraic closure with a valuation that extends val. In fact the tropicalization $\operatorname{trop} X$ does not change when we pass to $\overline{\mathbb{K}}$ (see [MS15, Theorem 3.2.4]).

Remark 2.3.11. Theorem 2.3.9 can also be formulated for any ideal I in $\mathbb{K}[x_1, \dots, x_n]$. In this case the set $\{\mathbf{w} \in \mathbb{R}^n : \operatorname{in}_{\mathbf{w}}(I) \neq (1)\}$ needs to be substituted with the set $\{\mathbf{w} \in \mathbb{R}^n : \operatorname{in}_{\mathbf{w}}(I) \text{ does not contain monomials}\}$.

Theorem 2.3.9 links the classical variety X to its tropicalization in a more explicit way. Moreover it is essential to define a polyhedral structure on $\operatorname{trop} X$.

Let f be a polynomial of degree d in $\mathbb{K}[x_1, \dots, x_n]$ then its homogenisation is the homogeneous polynomial $\tilde{f} = \sum c_{\mathbf{u}} x_0^{d-|\mathbf{u}|} \mathbf{x}^{\mathbf{u}} \in \mathbb{K}[x_0, \dots, x_n]$, where $|\mathbf{u}| = \sum_i u_i$. The homogenisation of an ideal $I \subset \mathbb{K}[x_1, \dots, x_n]$ is $I_{proj} = (\tilde{f} : f \in I)$.

Proposition 2.3.12. *[MS15, Proposition 3.2.8] Let* I *be an ideal of* $\mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ *and let* $X = V(I)$ *be its variety. Then* $\operatorname{trop}(X)$ *is a subcomplex of the Gröbner complex of* I_{proj} *thus it is the support of a* $\Gamma_{\mathrm{val}}-$*rational polyhedral complex in* $\mathbb{R}^n \cong \mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$.

The Gröbner complex is a polyhedral complex with the property that points in the relative interior of a cell produce the same initial ideal. Moreover points in distinct cells give rise to different initial ideals (see [MS15, Theorem 2.5.3]).

The polyhedral complex induced by the Gröbner complex on $\operatorname{trop} X$ has particular properties as the following theorem states:

Theorem 2.3.13. [Structure theorem for tropical varieties]
Let X *be an irreducible* $d-$*dimensional subvariety of* T^n*. Then* $\operatorname{trop}(X)$ *is the support of a balanced weighted* $\Gamma_{val}-$*rational polyhedral complex pure of dimension* d *and connected through codimension* 1.

If Σ is the polyhedral structure induced by the Gröbner complex the weight associated to a maximal cone σ is called the *multiplicity* of σ. It is defined by

$$\operatorname{mult}(\sigma) = \sum_{P \in Ass(\operatorname{in}_{\mathbf{w}}(I))} \operatorname{mult}(P, \operatorname{in}_{\mathbf{w}}(I)) \text{ for } \mathbf{w} \in \operatorname{relint}(\sigma) \tag{2.6}$$

where a prime ideal P is in $Ass(\operatorname{in}_{\mathbf{w}}(I))$ if it contains $\operatorname{in}_{\mathbf{w}}(I)$ and it is minimal with this property.

In the case of a hypersurface $V(f)$ in $\mathbb{T}^n$ the description of the polyhedral complex $\operatorname{trop} V(f)$ is more explicit.

Proposition 2.3.14. *[MS15, Proposition 3.1.6] Let $V(f) = V(\sum_{\mathbf{u} \in U \subset \mathbb{Z}^n} c_{\mathbf{u}} \mathbf{x}^{\mathbf{u}})$ be a hypersurface in T^n. The tropicalization $\operatorname{trop} V(f)$ is the $(n-1)$-skeleton of the polyhedral complex dual to the regular subdivision of the Newton polytope of f induced by the vector $(\operatorname{val}(c_{\mathbf{u}}))_{\mathbf{u} \in U}$.*

We can finally give the definition of *tropical variety* that we will use in this thesis.

Definition 2.3.15. A **tropical variety** in $\mathbb{R}^n$ is a balanced weighted polyhedral fan that is the tropicalization of a subvariety of T^n.

2.4 Tropical Grassmannian

The Grassmannian $\operatorname{Gr}(d,n)$ is the space parametrising d-dimensional linear spaces in $\mathbb{R}^n$. It can be embedded in $\mathbb{P}^{\binom{n}{d}-1}$ through the Plücker embedding. If we identify a point $q \in \operatorname{Gr}(d,n)$ with a linear space spanned by the rows of a $d \times n$ matrix then the Plücker map sends q to the $d \times d$ minors of this matrix. For this reason we denote the coordinates of $\mathbb{P}^{\binom{n}{d}-1}$ by $p_{i_1 \dots i_d}$ where $\{i_1, \dots, i_d\} \subset \{0, \dots, n-1\}$ and $|\{i_1, \dots, i_d\}| = d$. We identify $\operatorname{Gr}(d,n)$ with its image under the Plücker map. This is a subvariety of the projective space and we can consider $\operatorname{trop} \operatorname{Gr}(d,n) = \operatorname{trop}(\operatorname{Gr}(d,n) \cap T^{\binom{n}{d}-1})$.

We have the following result:

Theorem 2.4.1. *[SS04, Theorem 3.4] There is a bijection between the points in $\Gamma_{\mathrm{val}}^{\binom{n}{d}} \cap \operatorname{trop} \operatorname{Gr}(d,n)$ and the set of d-dimensional uniform tropical linear spaces in $\mathbb{R}^n / \mathbb{R}\mathbf{1}$.*

A tropical linear space is $\operatorname{trop} V(I)$ where $I \subset \mathbb{K}[x_0, \dots, x_{n-1}]$ is a homogeneous ideal generated by linear forms and $V(I) \subset \mathbb{P}^{n-1}$. It is uniform if the *matroid* associated to $V(I)$ is the uniform matroid. We will explain below the notion of a *matroid* and how to associate it to a linear subspace of $\mathbb{P}^{n-1}$.

In the following to simplify the notation we will consider $\mathbb{G}(d,n) := \mathrm{Gr}(d+1,n+1)$ that parametrizes linear spaces of dimension d in $\mathbb{P}^n$ and its tropicalization $\mathrm{trop}\,\mathbb{G}(d,n) = \mathrm{trop}(\mathrm{Gr}(d+1,n+1) \cap T^{\binom{n+1}{d+1}-1})$. The correspondence in Theorem 2.4.1 is now between points in $\Gamma_{\mathrm{val}} \cap \mathrm{trop}\,\mathbb{G}(d,n)$ and uniform tropical linear spaces in $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$.

Let L be a d-dimensional linear space in $\mathbb{P}^n$ and consider a matrix A_L whose row span is L.

Definition 2.4.2. Let E be the set $\{0,1,\dots,n\}$. The *matroid* M on E is a collection $\mathcal{C}$ of non-empty subsets $C \subset E$, called *circuits* of M that satisfies the following properties:

C1 No proper subset of a circuit is a circuit;

C2 If C_1 and C_2 are distinct circuits and $e \in C_1 \cap C_2$ then $C_1 \cup C_2 \smallsetminus \{e\}$ contains a circuit.

An *independent set* is a set in E that does not contain a circuit. A *basis* is a maximal independent set. All bases have the same cardinality. The *rank* of M is the cardinality of any basis. It is possible to define the matroid M using the set of bases and giving some axioms that substitute C1 and C2. A *flat* F of M is a subset of E such that $|C \smallsetminus F| \neq 1$ for all circuits C of M. The set of all flats is a partially ordered set called the *lattice of flats* $\mathcal{L}(M)$. Given two flats F_1 and F_2 we have that $F_1 \leq F_2$ if $F_1 \subset F_2$.

Example 2.4.3. Let $\mathcal{C}$ be the sets of all subsets of E with $d+2$ elements. This defines the collection of circuits a matroid on E called the uniform matroid $U_{d+1,n+1}$. Note a basis of $U_{d+1,n+1}$ is any subset of E with $d+1$ elements.

It is possible to associate to L a rank $d+1$ matroid M on the set $\{0,\dots,n\}$. The circuits of M will be given by the subsets $\{i_1,\dots,i_s\}$ such that the columns of A_L indexed by $i_1,\dots,i_s$ are a minimal linearly dependent subset of all the columns of A_L. A uniform tropical linear space is the tropicalization of a linear space whose associated matroid is $U_{d+1,n+1}$. This is equivalent to have all Pücker coordinates of L being nonzero.

The tropicalization $\mathrm{trop}\,M$ of a matroid M is defined to be the fan in $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ whose cones are $\mathrm{pos}(\mathbf{e}_{F_1},\dots,\mathbf{e}_{F_s}) + \mathbb{R}\mathbf{1}$ where $\varnothing \subset F_1 \subset F_2 \subset \ldots \subset F_s \subset E$ is a maximal chain in $\mathcal{L}(M)$ and $\mathbf{e}_{F_i} = \sum_{j \in F_i} \mathbf{e}_{j+1}$ (see [MS15, Theorem 4.2.6]).

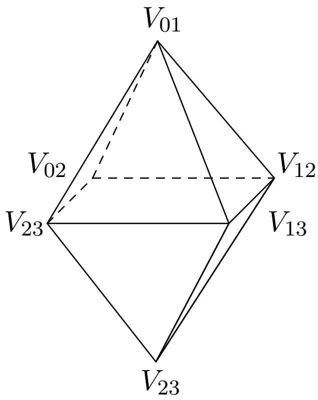

Figure 2.9: The matroid polytope associated to $U_{2,4}$ and the tropical line L as in Example 2.4.10

Proposition 2.4.4. *[MS15, Theorem 4.2.6] Let $\mathbb{K}$ be an algebraically closed field with trivial valuation. Then the support of* $\operatorname{trop} L$ *is the support of the fan* $\operatorname{trop} M$ *where M is the matroid associated to L.*

Example 2.4.5. Let $U_{2,3}$ be the uniform matroid on $E = \{0,1,2\}$. The set of flats is $\{\varnothing, \{0,1,2\}, \{0\}, \{1\}, \{2\}\}$. The tropicalization $\operatorname{trop} M$ is the fan whose cones are $\operatorname{pos}(\mathbf{e}_1)+\mathbb{R}\mathbf{1}, \operatorname{pos}(\mathbf{e}_2)+\mathbb{R}\mathbf{1}, \operatorname{pos}(\mathbf{e}_3)+\mathbb{R}\mathbf{1}$. If we quotient by $\mathbb{R}\mathbf{1}$ we obtain the standard tropical line as in Figure 2.8.

There is also another way of computing the tropicalization of L using the *matroid polytope*.

Definition 2.4.6. The *matroid polytope* P_M associated to a matroid M is defined by

$$P_M = \operatorname{conv}(\{\sum_{i \in B} \mathbf{e}_{i+1} =: \mathbf{e}_B \in \mathbb{R}^{n+1} : B \text{ is a basis of } M\}).$$

Example 2.4.7. Let $U_{2,3}$ be the uniform matroid on $E = \{0,1,2\}$. The matroid polytope is the convex hull of $(1,1,0), (1,0,1), (0,1,1)$ in $\mathbb{R}^3$.

Let $p = (p_{ij})$ be the point in $\operatorname{trop} \mathbb{G}(d,n)$ associated to $\operatorname{trop} L$. By Lemma 4.6 in [MS15] we have that p induces a regular subdivision Δ_p with the property that each face F of Δ_p is a matroid polytope $P_{M'}$. The bases of M' are $B_1, \dots, B_s$ where $F = \operatorname{conv}(\mathbf{e}_{B_1}, \dots, \mathbf{e}_{B_s})$.

Let L be a d-dimensional linear space in $\mathbb{P}^n$ with associated matroid M and assume that M is loop free, that is M has no circuits of size 1. Using the notations above we have:

Proposition 2.4.8. *The tropicalization* $\operatorname{trop} L$ *is the subcomplex of the dual complex* Σ *to* Δ_p *given by all points* $\boldsymbol{u} \in \Sigma$ *such that matroid* M' *associated to the dual face to* $\boldsymbol{u}$ *is loop free.*

The proposition gives us a way to compute the tropical linear space associated to a point p in $\operatorname{trop} \mathbb{G}(d, n)$.

Remark 2.4.9. If the matroid associated to L has some loops we have that L is contained in $\{[x_0 : \ldots : x_n] : x_i = 0 \text{ for } i \in I \subset \{0, 1, \ldots, n\}\}$ hence $\operatorname{trop} L = \operatorname{trop}(L \cap T^n) = \varnothing$. We will see that it is possible to extend the tropicalization to these type of linear spaces (see section 2.5). In chapter 4 we explain how to use Proposition 2.4.8 in this case.

Example 2.4.10. Let $(0, \ldots, 0) \in \operatorname{trop} \mathbb{G}(1, 3) \subset \mathbb{R}^6/\mathbb{R}\mathbf{1}$. We compute the tropical line $\operatorname{trop} L$ associated to it. By Theorem 2.4.1 we have that the matroid associated to L is the uniform matroid $U_{2,4}$. The matroid polytope is pictured in Figure 2.9 and the vertices are labelled by the bases of $U_{2,4}$. The regular subdivision Δ_p is the trivial one. The faces whose associated matroid has no loops are $\operatorname{conv}(V_{01}, V_{02}, V_{03}), \operatorname{conv}(V_{01}, V_{12}, V_{13}), \operatorname{conv}(V_{03}, V_{13}, V_{23}), \operatorname{conv}(V_{02}, V_{12}, V_{23})$.
The tropical line $\operatorname{trop} L$ is the fan in $\mathbb{R}^4/\mathbb{R}\mathbf{1}$ whose cones are $\operatorname{pos}(\mathbf{e}_1) + \mathbb{R}\mathbf{1}, \operatorname{pos}(\mathbf{e}_2) + \mathbb{R}\mathbf{1}, \operatorname{pos}(\mathbf{e}_3) + \mathbb{R}\mathbf{1}, \operatorname{pos}(\mathbf{e}_4) + \mathbb{R}\mathbf{1}$. If we consider the isomorphism in 2.5 we obtain the 1-dimensional fan in $\mathbb{R}^3$ whose cones are $\operatorname{pos}(\mathbf{e}_1), \operatorname{pos}(\mathbf{e}_2), \operatorname{pos}(\mathbf{e}_3), \operatorname{pos}(\mathbf{e}_0) = \operatorname{pos}(\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3)$. This is the analogue in $\mathbb{R}^3$ of the standard tropical line in $\mathbb{R}^2$.

2.5 Tropicalization II: toric varieties and their subvarieties

In this section we explain the tropicalization of toric varieties and we introduce the concept of *extended tropicalization* for projective varieties.

In the following we only consider normal toric varieties. For results on non-normal toric varieties see [CLS11, Appendix to Chapter 3].
Let T^n be the algebraic torus $(\mathbb{K}^*)^n$ with $M \cong \mathbb{Z}^n$ and $N = M^\vee = \operatorname{Hom}(M, \mathbb{Z}) \cong \mathbb{Z}^n$ the lattices of characters and one-parameter subgroups respectively.

Definition 2.5.1. A *toric variety* X is an irreducible variety with an action of the torus T^n and such that T^n is a dense subset of X.

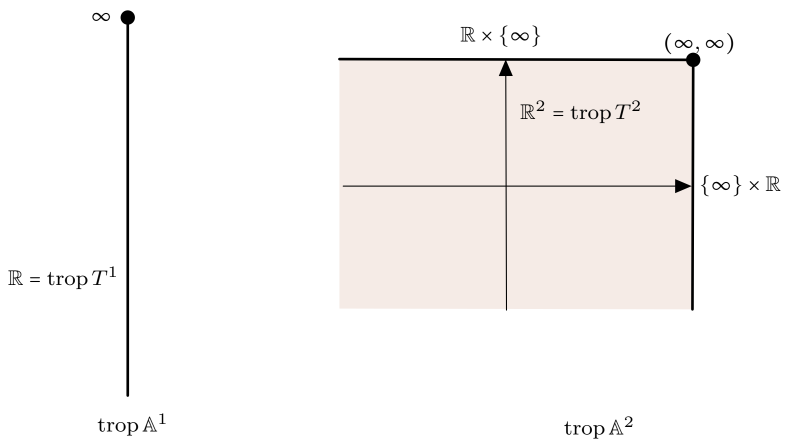

Figure 2.10: The tropicalization of $\mathbb{A}^1$ and $\mathbb{A}^2$

The action of the torus on X induces a stratification of X into torus orbits. The closure of each orbit is a subvariety of X. It is possible to associate to X a fan $\Sigma \subset \mathbb{R}^n \cong N \otimes \mathbb{R}$ that records all the torus orbits. There is a one to one correspondence between cones of Σ and torus orbits of X. Let σ be a cone of Σ, we denote by O_σ the corresponding torus orbit. Moreover we have that $\dim \overline{O}_\sigma = n - \dim \sigma$.

Any O_σ is isomorphic to a torus $T^{\dim O_\sigma}$ whose action is the restriction of the action of T^n to O_σ. This is the key fact that allows the definition of the tropicalization of X. In fact we define the tropicalization for each orbit and we glue all the pieces together.

We first explain the tropicalization for the simplest non trivial example of toric variety that is the affine line $\mathbb{A}^1$. The fan in $\mathbb{R}$ associated to $\mathbb{A}^1$ is given by the 1-dimensional cone $\operatorname{pos}(1)$ and the zero dimensional cone $\operatorname{pos}(0)$.

Using a generalization of part (3) of the Fundamental theorem of tropical geometry (Theorem 2.3.9) and assuming that val is not trivial we can define $\operatorname{trop}(\mathbb{A}^1)$ as $\operatorname{val}(\mathbb{A}^1) = \operatorname{val}(\mathbb{K}) = \mathbb{R} \cup \{\infty\}$. We observe that this is the union of $\operatorname{trop} T^1$ and $\operatorname{trop}(0) = \operatorname{val}(0) = \infty$. Moreover we can define a topology on $\operatorname{trop}(\mathbb{A}^1 \cap T^1)$ by taking as basis the usual open intervals and $(a, \infty]$ for any $a \in \mathbb{R}$. We obtain $\operatorname{trop} \mathbb{A}^1$ as the union of the tropicalization of the two torus orbits $\{0\}$ and $\{x \in \mathbb{A}^1 : x \neq 0\}$. In the same way we define the tropicalization of $\mathbb{A}^n$. This is given by $(\mathbb{R} \cup \{\infty\})^n$ (see Figure 2.10).

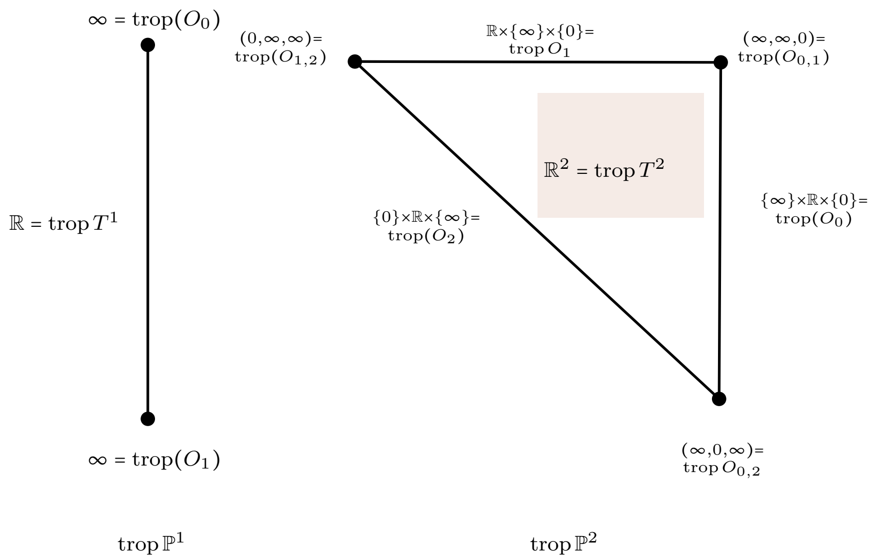

Figure 2.11: The tropicalization of $\mathbb{P}^1$ and $\mathbb{P}^2$

The following definition generalises the definition of tropicalization of $\mathbb{A}^n$ to any toric variety.

Definition 2.5.2. Let X_Σ be a toric variety and $\Sigma \subset \mathbb{R}^n$ where $U_\sigma^{\text{trop}} := \text{Hom}(\sigma^\vee \cap M, \mathbb{R} \cup \{\infty\})$ for $\sigma \in \Sigma$. The tropicalization of X_Σ is defined by

$$\operatorname{trop} X_\Sigma := \coprod_{\sigma \in \Sigma} U_\sigma^{\text{trop}}$$

and the topology is the pointwise-convergence topology that is induced from the product topology on $(\mathbb{R} \cup \{\infty\})^m$ with m the number of generators of $\sigma^\vee \cap M$.

Example 2.5.3. [The projective space $\mathbb{P}^n$] The second main example of toric variety is $\mathbb{P}^n$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the standard basis vectors for $\mathbb{R}^n$ and $\mathbf{e}_0 := -\sum_i \mathbf{e}_i$. The fan Σ associated to $\mathbb{P}^n$ is given by the cones spanned by any proper subset of $\{\mathbf{e}_0, \ldots, \mathbf{e}_n\}$. We will denote by $O_{i_1,\ldots,i_k}$ the orbit associated to the cone $\text{pos}(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_k})$. Note that this is the set $\{(x_0, \ldots, x_n) : x_j = 0 \text{ if and only if } j \in \{i_1, \ldots, i_k\}\}$.
The tropicalization $\operatorname{trop} \mathbb{P}^n$ is the union of $\binom{n+1}{k}$ copies of $\mathbb{R}^{n-k}$, one for each k-dimensional cone of Σ. It is obtained by gluing together $n+1$ copies of $(\mathbb{R} \cup \{\infty\})^n$ (see Figure 2.11).

The tropicalization of toric varieties and their subvarieties can be carried out using coordinates in a similar way to the subvarieties of T^n.

Let Σ be a fan in $N \otimes \mathbb{R} \cong \mathbb{R}^n$ with rays $\mathbf{r}_1, \dots, \mathbf{r}_s$, X_Σ be the corresponding toric variety and V be the matrix whose rows are the first lattice points on each $\mathbf{r}_i$. The ring $\mathbb{K}[x_1, \dots, x_s]$ is the Cox ring associated to X_Σ and its irrelevant ideal is $B = (\prod_{v_i \notin \sigma} x_i : \sigma \in \Sigma)$. The grading on $\mathbb{K}[x_1, \dots, x_s]$ is given by the map deg that is the map between $\mathbb{Z}^s$ and the class group $A_{n-1}(X_\Sigma)$ of X_Σ. This can be defined as the cokernel of the map f_V defined by V. In other words we have the following exact sequence:

$$0 \to M \cong \mathbb{Z}^n \xrightarrow{f_V} \mathbb{Z}^s \xrightarrow{\deg} A_{n-1}(X_\Sigma) \to 0. \tag{2.7}$$

If we apply the $\mathrm{Hom}(-, \mathbb{K}^*)$ functor we get:

$$0 \to H \to T^s \to T^n$$

where H is also a torus. The inclusion in T^s gives an action on $\mathbb{A}^s$. In particular H acts on $\mathbb{A}^s \smallsetminus V(B)$ and $X_\Sigma = (\mathbb{A}^s \smallsetminus V(B))/H$. The projection map is $\pi : \mathbb{A}^s \to X_\Sigma$. Whenever X_Σ is projective this quotient is the GIT quotient $\mathbb{A}^s//_\alpha H$ for some character α. This construction of X_Σ gives rise to the following result:

Proposition 2.5.4. *Let Σ be a simplicial fan. Then*

$$\mathrm{trop}(X_\Sigma) = (\mathrm{trop}(\mathbb{A}^s) \smallsetminus \mathrm{trop}\, V(B))/\,\mathrm{trop}\, H$$

where $\mathrm{trop}\, H$ *is the kernel of* V^T.

Example 2.5.5. We show that the construction of $\mathrm{trop}\,\mathbb{P}^1$ in Proposition 2.5.4 leads to the same tropicalization of Definition 2.5.2. The Cox ring of $\mathbb{P}^1$ is $\mathbb{K}[x_0, x_1]$, its irrelevant ideal is $B = (x_0, x_1)$ and $V^T = (1, -1)$. Hence

$$\mathrm{trop}\,\mathbb{P}^1 = \mathrm{trop}\,\mathbb{A}^2 \smallsetminus \mathrm{trop}\, V(B)/\,\mathrm{span}((1,1)) = (\mathbb{R} \cup \{\infty\})^2 \smallsetminus \{(\infty, \infty)\}/\,\mathrm{span}((1,1))$$

that is $\mathbb{R}$ union with two points $-\infty$ and ∞ that are the image under the quotient map of (x, ∞) and (∞, y) with $x, y \in \mathbb{R}$. These are two copies of $\mathrm{trop}\,\mathbb{A}^1 = (\mathbb{R} \cup \{\infty\})$ glued along $\mathbb{R}$. Note that the quotient map is an extension to $(\mathbb{R} \cup \{\infty\})^n$ of the map in (2.5).

The subvarieties of X_Σ can be described in a similar way to subvarieties of $\mathbb{P}^n$.

Proposition 2.5.6 (Proposition 5.2.4 [CLS11])**.** *Let Σ be a simplicial fan (i.e. the generators of each cone are linearly independent vectors). Then every subvariety*

$Y \subset X_\Sigma$ arises as $V(I) = \{\pi(x) \in X_\Sigma : f(x) = 0 \ \forall f \in I\}$ where I is a homogeneous ideal in the Cox ring with respect to the grading given by the map deg.

Remark 2.5.7. The hypothesis on Σ can be removed as shown in Proposition 5.2.8 of [CLS11] but the set $V(I)$ has to be defined in a different way. We are going to use this construction only for smooth toric varieties that are associated to simplicial fans.

We deduce that any subvariety Y of X_Σ is in correspondence with a subvariety Y' of $\mathbb{A}^n$. Moreover it is possible to define initial forms and initial ideals with respect to vectors in $(\mathbb{R} \cup \{\infty\})^n$ and reformulate Definitions 2.3.2 and 2.3.3 accordingly. Then by Corollary 6.2.16 in [MS15] the tropicalization of $Y' \subset X_\Sigma$ is equal to

$$((\bigcap_{f \in I} \operatorname{trop} V(f)) \smallsetminus \operatorname{trop} V(B))/\operatorname{trop} H \tag{2.8}$$

Given a torus T^n we can embed it in a toric variety X_Σ. Let Y be a subvariety of T^n. Then we can consider the closure $\overline{Y}$of Y in X_Σ. We can compute now the tropicalization of $\overline{Y}$ as subvariety of X_Σ. We call this the *extended tropicalization* of Y and we denote it by $\operatorname{trop}^{\text{ext}} Y$. The tropicalization $\operatorname{trop} Y$ as defined in Definitions 2.3.3 is contained in $\operatorname{trop}^{\text{ext}} Y$. More precisely we have :

Theorem 2.5.8. *[MS15, Theorem 6.2.18] Let $Y \subset T^n$ and $\overline{Y}$ the closure of Y in X_Σ. Then* $\operatorname{trop}^{\text{ext}} Y$ *is the closure of* $\operatorname{trop} Y \subset \mathbb{R}^n$ *in* $\operatorname{trop} X_\Sigma$.

The tropicalization $\operatorname{trop} Y$ gives information on $\overline{Y}$. The following result is Theorem 6.3.4 in [MS15] which is a reformulation of results in [Tev07].

Theorem 2.5.9. *Let $Y \subset T^n$ and $\overline{Y}$ the closure in X_Σ. For any cone $\sigma \in \Sigma$ we have $Y \cap O_\sigma \neq \varnothing$ if and only if the recession fan of* $\operatorname{trop} Y$ *intersects σ.*

Example 2.5.10. Consider $Y = V(x_0 + x_1 + x_2) \subset \mathbb{P}^2$. The extended tropicalization $\operatorname{trop}^{\text{ext}} Y$ is given by the standard tropical line with three points at *infinity.* These are the tropicalization of the points where Y intersects the orbits $O_{x_0}, O_{x_1}, O_{x_2}$ (see Figure 2.12).

Example 2.5.11. [The product of projective spaces] Another example of toric variety is $\mathbb{P}^1 \times \mathbb{P}^1$ or more in general $\mathbb{P}^{m_1} \times \ldots \times \mathbb{P}^{m_s}$. The associated fan is contained in $\mathbb{R}^{m_1 + \ldots + m_s}$ and it is the product of the fans of each projective space $\mathbb{P}^{m_i}$. For instance $\mathbb{P}^1 \times \mathbb{P}^1$ is associated to the fan whose cones are $\operatorname{pos}(\mathbf{e}_1, \mathbf{e}_2), \operatorname{pos}(-\mathbf{e}_1, -\mathbf{e}_2), \operatorname{pos}(\mathbf{e}_1, -\mathbf{e}_2)$, $\operatorname{pos}(\mathbf{e}_2, -\mathbf{e}_1)$. We denote the rays by $\mathbf{r}_1 = -\mathbf{r}_2 = \operatorname{pos}(\mathbf{e}_1)$ and $\mathbf{r}_3 = -\mathbf{r}_4 = \operatorname{pos}(\mathbf{e}_2)$.

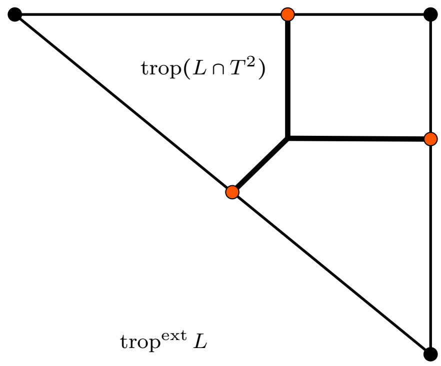

Figure 2.12: The tropicalization of the line $L = V(x_0 + x_1 + x_2) \subset \mathbb{P}^2$

We compute the tropicalization using 2.5.6. Let $\mathbb{K}[x_1, \dots, x_4]$ be the Cox ring, $B = (x_2x_4, x_1x_4, x_1x_3, x_2x_3)$ its irrelevant ideal and V the matrix

$$\begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}$$

whose kernel is generated by the span of the columns of the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$. The torus H is a two dimensional torus T^2 and it acts on $\mathbb{A}^4$ by

$$(t_1, t_2) \cdot (x_1, x_2, x_3, x_4) = (t_1x_1, t_1x_2, t_2x_3, t_2x_4).$$

We have that $\mathbb{P}^1 \times \mathbb{P}^1 = (\mathbb{A}^4 \setminus V(B))/H$. The tropicalization $\operatorname{trop}(\mathbb{P}^1 \times \mathbb{P}^1)$ is equal to $(\operatorname{trop} \mathbb{A}^4 \setminus \operatorname{trop} V(B))/\operatorname{trop} H$. The set $\operatorname{trop} V(B)$ is given by the points in $(\mathbb{R} \cup \{\infty\})^4$ where either the first two or the last two coordinates are infinity and $\operatorname{trop} H$ is the linear space spanned by $(1, 1, 0, 0)$ and $(0, 0, 1, 1)$. If we only look at $\operatorname{trop}((\mathbb{P}^1 \times \mathbb{P}^1) \cap T^2)$ we get the quotient of $\mathbb{R}^4$ with respect to the linear space spanned by $(1, 1, 0, 0)$ and $(0, 0, 1, 1)$. This is isomorphic to $\mathbb{R}^2$.
Let $f = x_1x_3 + x_2x_4$ be a polynomial in $\mathbb{K}[x_1, \dots, x_4]$. Consider the associated

hypersurface Y in $\mathbb{P}^1 \times \mathbb{P}^1$. The tropicalization of $Y \cap T^2$ is given by quotient by $\operatorname{trop} H$ of the linear space spanned by the vectors $(1,0,1,0), (1,1,0,0), (-1,0,0,1)$. This is the classical line spanned by the vector $(1,1)$ in $\mathbb{R}^2$.

We will compute another example of this tropicalization in section 3. In fact we will tropicalize the *flag varieties* which are subvarieties of a product of projective spaces.

Chapter 3

Toric degenerations via tropicalization

3.1 Introduction

Consider the variety $\mathcal{F}\ell_n$ of full flags $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = \mathbb{C}^n$ of vector subspaces of $\mathbb{C}^n$ with $\dim_{\mathbb{C}}(V_i) = i$. The flag variety $\mathcal{F}\ell_n$ is naturally embedded in a product of Grassmannians using the Plücker coordinates. We denote by I_n the defining ideal of $\mathcal{F}\ell_n$ with respect to this embedding. We produce toric degenerations of $\mathcal{F}\ell_n$ as Gröbner degenerations coming from the initial ideals associated to the maximal cones of $\operatorname{trop}\mathcal{F}\ell_n$. Moreover, we compare these with certain toric degenerations arising from representation theory.

We will consider 1-parameter toric degenerations of $\mathcal{F}\ell_n$. These are flat families $\varphi : \mathcal{F} \to \mathbb{A}^1$, where the fibre over zero (also called *special* fibre) is a toric variety and all other fibres are isomorphic to $\mathcal{F}\ell_n$. Once we have such a degeneration, some of the algebraic invariants of $\mathcal{F}\ell_n$ will be the same for all fibres, hence the computations can be done on the toric fibre. In the case of a toric variety such invariants are easier to compute than in the case of a general variety. In fact, they have a nice combinatorial description. Moreover, toric degenerations connect different areas of mathematics, such as symplectic geometry, representation theory, and algebraic geometry.

Let $X = V(I)$ be a projective variety and $\operatorname{trop} X$ be its tropicalization. The initial ideals associated to the top-dimensional cones of $\operatorname{trop} X$ are good candidates to give toric degenerations (see Lemma 3.2.2). For example, in the case of Grassmannians $\operatorname{Gr}(2,n)$ each maximal cone of $\operatorname{trop}\operatorname{Gr}(2,n)$ gives a toric degeneration, see [SS04; Wit15; BFG+16]. However, this is not true for the Grassmannians $\operatorname{Gr}(3,n)$.

In [MS] Mohammadi and Shaw identify which maximal cones of $\operatorname{trop}\operatorname{Gr}(3,n)$ produce such degenerations.

The following are our main results. More detailed formulations can be found in Theorem 3.3.3, Theorem 3.3.5, and Proposition 3.5.3. We will call a maximal cone C of $\operatorname{trop} X$ *prime* if $\operatorname{in}_C(I) := \operatorname{in}_{\mathbf{w}}(I)$ is prime, with $\mathbf{w}$ a vector in the relative interior of C.

Theorem 7. *The tropical variety* $\operatorname{trop}\mathcal{F}\ell_4 \subset \mathbb{R}^{14}/\mathbb{R}^3$ *is a 6-dimensional fan with* 78 *maximal cones. From prime cones we obtain four non-isomorphic toric degenerations. After applying Procedure 1 we obtain at least two additional non-isomorphic toric degenerations from non-prime cones.*

Theorem 8. *The tropical variety* $\operatorname{trop}\mathcal{F}\ell_5 \subset \mathbb{R}^{30}/\mathbb{R}^4$ *is a 10-dimensional fan with* 69780 *maximal cones. From prime cones we obtain 180 non-isomorphic toric degenerations.*

Toric degenerations of flag varieties and Schubert varieties have been studied intensively in representation theory over the last two decades. We refer the reader to [FFL16a] for an overview on this topic and to the references therein.

The main motivation of this work is to study the flat degenerations of flag varieties into toric varieties arising from the tropicalization and to compare these degenerations to those associated to *string polytopes* and the *Feigin-Fourier-Littelmann-Vinberg polytope (FFLV polytope).*

Theorem 3.1.1. *For $\mathcal{F}\ell_4$ there is at least one new toric degeneration arising from prime cones of* $\operatorname{trop}\mathcal{F}\ell_4$ *in comparison to those obtained from string polytopes and the FFLV polytope.*

For $\mathcal{F}\ell_5$ there are at least 168 new toric degenerations arising from prime cones of $\operatorname{trop}\mathcal{F}\ell_5$ *in comparison to those obtained from string polytopes and the FFLV polytope.*

Our work is closely related to the theory of Newton–Okounkov bodies. Let $\mathbb{K}$ be a not necessarily algebraically closed field and X a projective variety. It is possible to associate to every prime cone in $\operatorname{trop} X$ a valuation with a finite *Khovanskii basis* B on the homogeneous coordinate ring $\mathbb{K}[X]$, see [KM16, Lemma 5.7]. This is a set of elements of $\mathbb{K}[X]$, such that their valuations generate the value semigroup $S(\mathbb{K}[X],\mathrm{val})$. The convex hull of $S(\mathbb{K}[X],\mathrm{val}) \cup \{0\}$ is referred to as the *Newton–Okounkov cone*. After intersecting this cone with a particular hyperplane one obtains a convex body, called the *Newton–Okounkov body.* When a finite Khovanskii basis exists, [And13, Theorem 1.1] states that there is a flat degeneration of the variety

X into a toric variety whose normalization has as associated polytope the Newton–Okounkov body. In this case the Newton–Okounkov body is a polytope. The toric polytopes obtained in Theorem 3.3.3, Theorem 3.3.5, and Proposition 3.5.3 can be seen as Newton–Okounkov bodies for the valuations defined in §6.

The chapter is structured as follows. In §3.2 we provide the necessary background. We study the tropicalization of the flag varieties $\mathcal{F}\ell_n$ for $n = 4, 5$ and the induced toric degenerations in §3.3. The solutions to [Stu17, Problem 5 on Grassmannians] and [Stu17, Problem 6 on Grassmannians] can be found in Theorem 3.3.3.

In §3.4 we compare the toric degenerations found using the tropicalization to the toric degenerations associated to the *string polytopes* and the *Feigin-Fourier-Littelmann-Vinberg polytope (FFLV polytope).*

In §3.5 we give an algorithmic approach to solving [KM16, Problem 1] for a subvariety X of a toric variety Y when each cone in $\operatorname{trop} X$ has multiplicity one. Procedure 1 aims at computing a new embedding X' of X in case $\operatorname{trop} X$ has some non-prime cones. Once we have such an embedding, we explain how to get new toric degenerations of X. We apply the procedure to $\mathcal{F}\ell_4$. Furthermore, we explain how to interpret the procedure in terms of finding valuations with finite Khovanskii basis on the algebra given by the homogeneous coordinate ring of X.

3.2 Preliminary notions

In this section we recall the definition of flag variety and the necessary background in tropical geometry. In fact, the key ingredient in the study of Gröbner toric degenerations of $\mathcal{F}\ell_n$ is the subfan of the Gröbner fan of I_n given by $\operatorname{trop} \mathcal{F}\ell_n$.

Let $\mathbb{K}$ be a field with $\operatorname{char}(\mathbb{K}) = 0$ and consider on it the trivial valuation. We are mainly interested in the case $\mathbb{K} = \mathbb{C}$.

Definition 3.2.1. A *complete flag* in the vector space $\mathbb{K}^n$ is a chain

$$\mathcal{V}: \ \{0\} = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = \mathbb{K}^n$$

of vector subspaces of $\mathbb{K}^n$ with $\dim_{\mathbb{K}}(V_i) = i$.

The set of all complete flags in $\mathbb{K}^n$ is denoted by $\mathcal{F}\ell_n$ and it has an algebraic variety structure. More precisely, it is a subvariety of the product of Grassmannians $\operatorname{Gr}(1, n) \times \operatorname{Gr}(2, n) \times \cdots \times \operatorname{Gr}(n-1, n)$.

Composing with the Plücker embedding of the Grassmannians, $\mathcal{F}\ell_n$ becomes a subvariety of $\mathbb{P}^{\binom{n}{1}-1} \times \mathbb{P}^{\binom{n}{2}-1} \times \cdots \times \mathbb{P}^{\binom{n}{n-1}-1}$ and so we can ask for its defining ideal

I_n. Each point in the flag variety can be represented by an $(n-1)\times n$-matrix $M=(x_{i,j})$ whose first d rows generate V_d. Each V_d corresponds to a point in $\mathrm{Gr}(d,n)$. Moreover, they satisfy the condition $V_d\subset V_{d+1}$ for $d=0,\dots,n-1$. In order to compute the ideal I_n defining $\mathcal{F}\ell_n$ in $\mathbb{P}^{\binom{n}{1}-1}\times\mathbb{P}^{\binom{n}{2}-1}\times\cdots\times\mathbb{P}^{\binom{n}{n-1}-1}$ we have to translate the inclusions $V_d\subset V_{d+1}$ into polynomial equations. We define the map

$$\varphi_n:\ \mathbb{K}[p_J:\ \varnothing\neq J\subsetneq\{0,\dots,n-1\}]\to\mathbb{K}[x_{i,j}:\ 0\le i\le n-2,\ 1\le j\le n-1]$$

sending each Plücker variable p_J to the determinant of the submatrix of M with row indices $1,\dots,|J|$ and column indices in J. The ideal I_n of $\mathcal{F}\ell_n$ is the kernel of φ_n. Consider an inner product on V. Then there is an action of $S_n\rtimes\mathbb{Z}_2$ on $\mathcal{F}\ell_n$. The symmetric group acts by permuting the columns of M. The action of $\mathbb{Z}_2$ maps a complete flag $\mathcal{V}$ to its complement, which is defined to be

$$\mathcal{V}^\perp:\ \{0\}=V_n^\perp\subset V_{n-1}^\perp\subset\cdots\subset V_1^\perp\subset V_0^\perp=\mathbb{K}^n.$$

We will hence do computations up to $S_n\rtimes\mathbb{Z}_2$-symmetry. We are interested in finding toric degenerations. These are degenerations whose special fibre is a toric variety hence it is defined by a *toric* ideal, i.e. a binomial prime ideal not containing monomials ([CLS11, Lemma 1.1.17]). This toric ideal arises as initial ideal of I_n.

We now recall some properties of the initial ideals.

Let $S=\mathbb{K}[x_0,\dots,x_n]$ and I be a homogeneous ideal in S. An important geometric property of initial ideals is that there exists a flat family over $\mathbb{A}^1$ for which the fibre over 0 is isomorphic to $V(\mathrm{in}_{\mathbf{w}}(I))$ and all the other fibres are isomorphic to the variety $V(I)$. Here, if J is a homogeneous ideal of S then we define $V(J):=\mathrm{Proj}(S/J)$ where the grading on S and hence on S/J comes from the ambient space which has S as homogeneous coordinate ring.

Let t be the coordinate in $\mathbb{A}^1$, then the flat family is given by the ideal

$$\tilde{I}_t=\langle t^{-\min_{\mathbf{u}}\{\mathbf{w}\cdot\mathbf{u}\}}f(t^{w_0}x_0,\dots,t^{w_n}x_n):\ f=\sum a_{\mathbf{u}}x^{\mathbf{u}}\ \ in\ I\rangle\subset\mathbb{K}[t,x_0,\dots,x_n].$$

This family gives a flat degeneration of the variety $V(I)$ into the variety $V(\mathrm{in}_{\mathbf{w}}(I))$ called the *Gröbner degeneration.* In order to look for toric degenerations, we study the tropicalization of $V(I)$.

Let $V(I)$ be a $(d-1)$-dimensional irreducible subvariety of $\mathbb{P}^n$ that intersects the torus $T^n=(\mathbb{K}^*)^{n+1}/\mathbb{K}^*$ non-trivially. We consider the tropicalization $\mathrm{trop}\,V(I)=\mathrm{trop}\,V(I)\cap T^n$. This is the support of a rational fan given by the quotient by $\mathbb{R}\mathbf{1}$ of a subfan F of the Gröbner fan of I (Theorem 2.3.13). The fan F has dimension d, which is the Krull dimension of S/I. If we consider this fan structure on $\operatorname{trop} V(I)$ we have that vectors in the relative interior of a cone give rise to the same initial ideal and vectors in distinct relative cone interiors induce distinct initial ideals (see section 2.3). For this reason we denote by $\operatorname{in}_C(I)$ the initial ideal of I with respect to any $\mathbf{w}$ in the relative interior of a cone C.

Consider now S to be the Cox ring of $\mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_s}$. The ring S has a $\mathbb{Z}^s$-grading given by $\deg : \mathbb{Z}^{n+1} \to \mathbb{Z}^s$ that is the map in the short exact sequence (2.7). An ideal I defining an irreducible subvariety $V(I)$ of $\mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_s}$ is homogeneous with respect to this grading (Proposition 2.5.6). We consider the tropicalization of $V(I) \cap T^{m_1+\ldots+m_s}$. As we have seen in section 2.5 this is contained in $\mathbb{R}^{m_1+\ldots+m_s+s}/H$, where H is an s-dimensional linear space that is the image of deg. Similarly to the projective case, if $V(I)$ is a d-dimensional irreducible subvariety of $\mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_s}$, then $\operatorname{trop} V(I)$ is the support of a fan, which is the quotient by H of a rational $(d+s)$-dimensional subfan F of the Gröbner fan of I. Here the Krull dimension of S/I is $d+s$.

In the following we will always consider $\operatorname{trop} V(I)$ with the fan structure defined above. Each cone of $\operatorname{trop} V(I)$ corresponds to an initial ideal which contains no monomials. Moreover the good candidates for toric degenerations are the initial ideals corresponding to the relative interior of the maximal cones (see Lemma 3.2.2 below). We say a maximal cone is *prime* if the corresponding initial ideal is prime.

The ring S will denote either the Cox ring of $\mathbb{P}^n$ or of $\mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_s}$. The following results can be proved for any Cox ring S of a toric variety X_Σ but we restrict to the case $\mathbb{P}^n$ or of $\mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_s}$ since we will only need these for our purposes.

Lemma 3.2.2. *Let $I \subset S$ be a homogeneous ideal and C a cone of* $\operatorname{trop} V(I)$.

1. *If* $\operatorname{in}_C(I)$ *is binomial then C is maximal;*
2. *If* $\operatorname{in}_C(I)$ *is a toric ideal, i.e. binomial and prime, then C is maximal and it has multiplicity one;*
3. *If C is a maximal cone of multiplicity one, then* $\operatorname{in}_C(I)$ *has a unique toric ideal in its primary decomposition.*

Proof. Suppose C is not maximal. Then there exists a maximal cone $C' \in \operatorname{trop} V(I)$ such that C is a face of C'. Let $\mathbf{w}$ be in C we have that $\operatorname{in}_{C'}(I) = \operatorname{in}_{\mathbf{v}+\epsilon\mathbf{w}}(I)$ where

$\mathbf{v}+\epsilon\mathbf{w}$ is in C'. By Lemma 2.4.6 in [MS15] we have $\mathrm{in}_{C'}(I) = \mathrm{in}_{\mathbf{v}+\epsilon\mathbf{w}}(I) = \mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I))$. The ideal $\mathrm{in}_C(I)$ is binomial hence its initial ideals are either equal to $\mathrm{in}_C(I)$ or they contain some monomials. In both case we conclude that $C = C'$ hence C is maximal. This proves (1).
We first prove (2) and (3) for S the Cox ring of $\mathbb{P}^n$. Let $I' = \mathrm{in}_C(I)\mathbb{K}[x_1^{\pm1},\dots,x_n^{\pm1}]$ and consider the subvariety $V(I')$ of the torus T^n. Then by [MS15, Remark 3.4.4] the multiplicity of a maximal cone C is counting the number of d-dimensional torus orbits whose union is $V(I')$. Since $\mathrm{in}_C(I)$ is toric, then $V(I')$ is an irreducible toric variety having a unique d-dimensional torus orbit. Hence C has multiplicity one.

Suppose now C has multiplicity one. This implies that $\mathrm{in}_C(I)$ contains one associated prime J which does not contain monomials. The ideal J has to be binomial since it is the ideal of the closure of the unique d-dimensional torus orbit contained in $V(I')$.

When S is the total coordinate ring of $\mathbb{P}^{m_1}\times\cdots\times\mathbb{P}^{m_s}$, the torus is given by $T^{m_1}\times\cdots\times T^{m_s}\cong T^{m_1+\cdots+m_s}$. We may assume that for each i,

$$T^{m_i} = \{[1 : a_1 : \ldots : a_{m_i}] \in \mathbb{P}^{m_i} : \ a_j \neq 0 \text{ for all } j\}.$$

The variables for $\mathbb{P}^{m_i}$ are denoted by $x_{i,0},\dots,x_{i,m_i}$ for each i. We fix the Laurent polynomial ring

$$S' = \mathbb{K}[x_{1,1}^{\pm1},\dots,x_{1,m_1}^{\pm1},x_{2,1}^{\pm1},\dots,x_{2,m_2}^{\pm1},\dots,x_{s,1}^{\pm1},\dots,x_{s,m_s}^{\pm1}].$$

We consider the ideal $I' = \mathrm{in}_C(I)S'$ in S' and the variety $V(I')$ as a subvariety of $T^{m_1+\ldots+m_s}$. Then the proof proceeds as before. □

Remark 3.2.3. From Lemma 3.2.2 we conclude the multiplicity being one is a necessary but not sufficient condition for toric initial ideals. A maximal cone can have multiplicity one but its associated initial ideal might be neither prime nor binomial. There may be associated primes that contain monomials in the decomposition of $\mathrm{in}_{\mathbf{w}}(I)$ and these do not contribute to the multiplicity. We list examples of such cones in $\mathrm{trop}\,\mathcal{F}\ell_5$ as we will see in Theorem 3.3.5.

We now describe some properties of the toric initial ideals corresponding to maximal cones of $\mathrm{trop}\,V(I)$. Let C be a cone in $\mathrm{trop}\,V(I)$ and $\{\mathbf{w}_1,\dots,\mathbf{w}_d\}$ be d linearly independent vectors in F generating the maximal cone C', such that $C'/H\cong C$. We can assume that the $\mathbf{w}_i$'s have integer entries since F is a $\mathbb{Q}$–rational fan. We define the matrix associated to C to be

$$W_C = [\mathbf{w}_1, \dots, \mathbf{w}_d]^T. \tag{3.1}$$

Consider a sublattice L of $\mathbb{Z}^{n+1}$ and the standard basis $\mathbf{e}_1, \dots, \mathbf{e}_{n+1}$ of $\mathbb{Z}^{n+1}$. Given $\ell = (\ell_1, \dots, \ell_{n+1}) \in L$ we set $\ell^+ = \sum_{\ell_i > 0} \ell_i \mathbf{e}_i$ and $\ell^- = -\sum_{\ell_j < 0} \ell_j \mathbf{e}_j$. Note that $\ell = \ell^+ - \ell^-$ and $\ell^+, \ell^- \in \mathbb{N}^{n+1}$. We use here the same notation of [CLS11, page 15].

Lemma 3.2.4. *Let I be a homogeneous ideal in S and C a maximal cone in* $\operatorname{trop} V(I)$. *If* $\operatorname{in}_C(I)$ *is toric, then there exists a sublattice L of $\mathbb{Z}^{n+1}$ and constants $0 \neq c_\ell \in \mathbb{K}$ with $\ell \in L$ such that*

$$\operatorname{in}_C(I) = I(W_C) := \langle \mathbf{x}^{\ell^+} - c_\ell \mathbf{x}^{\ell^-} : \ \ell \in L \rangle.$$

In particular, L is the kernel of the map $f : \mathbb{Z}^{n+1} \to \mathbb{Z}^d$ defined by the matrix W_C. If C has multiplicity one and $\operatorname{in}_C(I)$ *is not toric, then the unique toric ideal in the primary decomposition of* $\operatorname{in}_C(I)$ *is of the form $I(W_C)$.*

Proof. Let $\operatorname{in}_C(I) \subset S$ be a toric initial ideal and let C' be the corresponding cone in F. For every $\mathbf{w}'$ and $\mathbf{w}$ in the relative interior of C' we have $\operatorname{in}_{\mathbf{w}'}(I) = \operatorname{in}_C(I) = \operatorname{in}_{\mathbf{w}}(I)$. This implies $\operatorname{in}_C(I)$ is W_C-homogeneous with respect to the $\mathbb{Z}^d$-grading on S given by the matrix W_C. By [Stu96, Lemma 10.12] there exists an automorphism ϕ of S sending x_i to $\lambda_i x_i$ for some $\lambda_i \in \mathbb{K}$, such that the ideal $\operatorname{in}_C(I)$ is isomorphic to an ideal of the form

$$I_L := \langle \mathbf{x}^{\ell^+} - \mathbf{x}^{\ell^-} : \ \ell \in L \rangle.$$

Here L is the sublattice of $\mathbb{Z}^{n+1}$ given by the kernel of the map $f : \mathbb{Z}^{n+1} \to \mathbb{Z}^d$. Applying ϕ^{-1} to $\operatorname{in}_C(I)$ we can write each toric initial ideal as

$$\langle \mathbf{x}^{\ell^+} - c_\ell \mathbf{x}^{\ell^-} : \ \ell \in L \rangle = I(W_C),$$

for some $0 \neq c_\ell \in \mathbb{K}$, L and W_C defined above.

Let C be a cone of multiplicity one and suppose $\operatorname{in}_C(I)$ is not prime. Then by Lemma 3.2.2 there exists a unique toric ideal J in the primary decomposition of $\operatorname{in}_C(I)$. This toric ideal J contains $\operatorname{in}_C(I)$ and we will show that it can be expressed as $I(W_C)$. The variety $V(I)$ is considered as subvariety of $\mathbb{P}^n$. As in Lemma 3.2.2, the case $V(I) \subset \mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_s}$ has an analogous proof.

The tropical variety depends only on the intersection of $V(I)$ with the torus, and $\operatorname{in}_C(I)\mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is equal to J. Hence, J is a prime ideal that is homogeneous with respect to W_C so we can proceed as above to show J can be written as

$\langle \mathbf{x}^{\ell^+} - c_\ell \mathbf{x}^{\ell^-} : \ell \in L \rangle = I(W_C)$.

□

Remark 3.2.5. Note that the lattice L and the ideal $I(W_C)$ only depend on the linear space spanned by the rays of the cone C'. Hence they are the same for every set of d independent vectors in C' chosen to define W_C.

3.3 Tropicalization and toric degenerations

In this section we study the tropicalization of $\mathcal{F}\ell_4$ and $\mathcal{F}\ell_5$. We analyze the Gröbner toric degenerations arising from $\operatorname{trop}\mathcal{F}\ell_4$ and $\operatorname{trop}\mathcal{F}\ell_5$, and we compute the polytopes associated to their normalizations. In Proposition 3.3.4 we describe the *tropical configurations* arising from the maximal cones of $\operatorname{trop}\mathcal{F}\ell_4$. These are configurations of a point on a tropical line in a tropical plane corresponding to the points in the relative interior of a maximal cone.

Before stating our main results, we recall the following definition.

Definition 3.3.1. There exists a *unimodular equivalence* between two lattice polytopes P and Q (resp. two fans $\mathcal{F}$ and $\mathcal{G}$) if there exists an affine lattice isomorphism ϕ of the ambient lattices sending the vertices (resp. the rays) of one polytope (resp. fan) to the vertices (resp. rays) of the other. Moreover, if σ is a face of P (resp. of $\mathcal{F}$) then $\phi(\sigma)$ is a face of Q (resp. $\mathcal{G}$) and the adjacency of faces is respected.

Remark 3.3.2. We are interested in finding distinct fans up to unimodular equivalence as they give rise to non-isomorphic toric varieties. Often it will be possible only to determine combinatorial equivalence (see [CLS11§2.2]). This is a weaker condition but when it does not hold it allows us to rule out unimodular equivalence.

Theorem 3.3.3. *The tropical variety* $\operatorname{trop}\mathcal{F}\ell_4$ *is a* 6*-dimensional rational fan in* $\mathbb{R}^{14}/\mathbb{R}^3$ *with a* 3*-dimensional lineality space. It consists of 78 maximal cones, 72 of which are prime. They are organized in five* $S_4 \rtimes \mathbb{Z}_2$*-orbits, four of which contain prime cones. The prime cones give rise to four non-isomorphic toric degenerations.*

Proof. The theorem is proved by explicit computations. We developed a `Macaulay2`[GS] package called `ToricDegenerations` containing all the functions we use. The package and the data needed for this proof are available at

`https://github.com/ToricDegenerations`.

The flag variety $\mathcal{F}\ell_4$ is a 6-dimensional subvariety of $\mathbb{P}^3 \times \mathbb{P}^5 \times \mathbb{P}^3$. The ideal I_4 defined in the previous section is contained in the Cox ring R of $\mathbb{P}^3 \times \mathbb{P}^5 \times \mathbb{P}^3$ which is the polynomial ring over $\mathbb{C}$ on the variables

$$p_0, p_1, p_2, p_3, p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23}, p_{012}, p_{013}, p_{023}, p_{123}.$$

The grading on R is given by the matrix

$$D = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}. \tag{3.2}$$

The explicit form of I_4 can be found in [MS05, page 276]. As we have seen in §3.2 the tropicalization of $\mathcal{F}\ell_4$ is contained in $\mathbb{R}^{14}/H$. In this case H is the vector space spanned by the rows of D.

We use the `Macaulay2` interface to `Gfan` [Jen] to compute $\operatorname{trop} \mathcal{F}\ell_4$. The given input is the ideal I_4 and the $S_4 \rtimes \mathbb{Z}_2$-action. The output is a subfan F of the Gröbner fan of dimension 9. We quotient it by H to get $\operatorname{trop} \mathcal{F}\ell_4$ as a 6-dimensional fan contained in $\mathbb{R}^{14}/H \cong \mathbb{R}^{14}/\mathbb{R}^3$.

Firstly, the function `computeWeightVectors` computes a list of vectors. There is one for every maximal cone of $\operatorname{trop} \mathcal{F}\ell_4$ and it is contained in the relative interior of the corresponding cone. Then `groebnerToricDegenerations` computes all the initial ideals and checks if they are binomial and prime over $\mathbb{Q}$. These are organized in a hash table, which is the output of the function. All 78 initial ideals are binomial and all maximal cones have multiplicity one. In order to check primeness over $\mathbb{C}$, we check if $\operatorname{in}_C(I_4) = I(W_C)$. By Lemma 3.2.4 we have that the generators of $I(W_C)$ are strictly related to the generators of I_L hence comparing these we are able to show whether $\operatorname{in}_C(I_4) = I(W_C)$.

We consider the orbits of the $S_4 \ltimes \mathbb{Z}_2$-action on the set of initial ideals. These correspond to the orbits of maximal cones of F and hence of $\operatorname{trop} \mathcal{F}\ell_4$. There is one orbit of non-prime initial ideals and four orbits of prime initial ideals. The varieties corresponding to initial ideals contained in the same orbit are isomorphic. Therefore, for each orbit we choose a representative of the form $\operatorname{in}_C(I_4) = I(W_C)$ for some cone C.

We now compute for each of the four prime orbits, the polytope of the normalization of the associated toric varieties. We use the `Macaulay2`-package `Polyhedra` [Bir] for the following computations.

The lattice M associated to $S/I(W_C)$ is generated over $\mathbb{Z}$ by the columns of

W_C. To use `Polyhedra` we want to have a lattice with index 1 in $\mathbb{Z}^9$. Hence, in case the index of M in $\mathbb{Z}^9$ is different from 1, we consider M as the lattice generated by the columns of the matrix $(\ker((\ker(W_C))^T)^T$. Here, for a matrix A we consider $\ker(A)$ to be the matrix whose columns minimally generate the kernel of the map $\mathbb{Z}^{14} \to \mathbb{Z}^9$ defined by A. We denote the set of generators of M by $\mathcal{B}_C = \{\mathbf{b}_1, \dots, \mathbf{b}_{14}\}$ so that $M = \mathbb{Z}\mathcal{B}_C$.

The toric variety $\mathbb{P}^3 \times \mathbb{P}^5 \times \mathbb{P}^3$ can be seen as $\mathrm{Proj}(\oplus_\ell R_{\ell(1,1,1)})$ and $I(W_C)$ as an ideal in $\oplus_\ell R_{\ell(1,1,1)}$ (see [MS05, Chapter 10]). The associated toric variety is $\mathrm{Proj}(\oplus_\ell \mathbb{C}[\mathbb{N}\mathcal{B}_C]_{\ell(1,1,1)})$. The polytope P of the normalization is given as the convex hull of those lattice points in $\mathbb{N}\mathcal{B}_C$ corresponding to degree $(1,1,1)$-monomials in $\mathbb{C}[\mathbb{N}\mathcal{B}_C]$.

These can be found in the following way. We order the rows of the matrix $(\mathbf{b}_1, \dots, \mathbf{b}_{14})$ associated to $\mathcal{B}_C$ so that the first three rows give the matrix D from (3.2). Now the matrix $(\mathbf{b}_1, \dots, \mathbf{b}_{14})$ represents a map $\mathbb{Z}^{14} \to \mathbb{Z}^3 \oplus \mathbb{Z}^6$, where $\mathbb{Z}^3 \oplus \mathbb{Z}^6$ is the lattice M and the $\mathbb{Z}^3$ part gives the degree of the monomials associated to each lattice point on M. The lattice points, whose convex hull give the polytope P, are those ones with the first three coordinates being 1. In other words, we have obtained P by applying the reverse procedure of constructing a toric variety from a polytope (see [CLS11§2.1-§2.2]). Note that the difference from the procedure given in [CLS11§2.1-§2.2] is the $\mathbb{Z}^3$-grading and because of that we do not consider the convex hull of $\mathcal{B}_C$, but the intersection of $\mathbb{N}\mathcal{B}_C$ with the hyperplane $y_1 = y_2 = y_3 = 1$ in $\mathbb{R}^9 \supset \mathbb{N}\mathcal{B}_C$.

In Table 3.1 there are the numerical invariants of the initial ideals and their corresponding polytopes. Using `polymake` [GJ] we first obtain that there is no combinatorial equivalence between each pair of polytopes. This means that there is no unimodular equivalence between the corresponding normal fans, hence the normalization of the toric varieties associated to these toric degenerations are not isomorphic. This implies that we obtain four non-isomorphic toric degenerations. □

Proposition 3.3.4. *There are six tropical configurations up to symmetry (depicted in Figure 3.2) arising from the maximal cones of* $\mathrm{trop}\,\mathcal{F}\ell_4$. *They are further organized in five* $S_4 \rtimes \mathbb{Z}_2$*-orbits.*

Proof. The tropical variety $\mathrm{trop}\,\mathcal{F}\ell_4$ is contained in

$$\mathrm{trop}\,\mathrm{Gr}(1,4) \times \mathrm{trop}\,\mathrm{Gr}(2,4) \times \mathrm{trop}\,\mathrm{Gr}(3,4).$$

Orbit	Size	Cohen-Macaulay	Prime	#Generators	F-vector of associated polytope
1	24	Yes	Yes	10	(42, 141, 202, 153, 63, 13)
2	12	Yes	Yes	10	(40, 132, 186, 139, 57, 12)
3	12	Yes	Yes	10	(42, 141, 202, 153, 63, 13)
4	24	Yes	Yes	10	(43, 146, 212, 163, 68, 14)
5	6	Yes	No	10	Not applicable

Table 3.1: The tropical variety $\operatorname{trop} \mathcal{F}\ell_4$ has 78 maximal cones organized in five $S_4 \rtimes \mathbb{Z}_2$-orbits. The algebraic invariants of the initial ideals associated to these cones and the F-vectors of their associated polytopes are listed here.

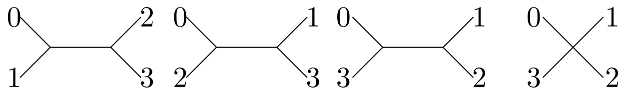

Figure 3.1: Combinatorial types of tropical lines in $\mathbb{R}^4/\mathbb{R}\mathbf{1}$.

Each tropical Grassmannian parametrizes tropicalized linear spaces (see section 2.4). This implies that every point p in $\operatorname{trop} \mathcal{F}\ell_4$ corresponds to a chain of tropical linear subspaces given by a point on a tropical line contained in a tropical plane. All tropical chains are *realizable*, meaning that they are the tropicalization of the classical chains of linear spaces of $\mathbb{K}^4$ corresponding to a point q in $\mathcal{F}\ell_4$ such that $\operatorname{val}(q) = p$, where $\mathbb{K} = \mathbb{C}\{\{t\}\}$ and val is the natural valuation on this field.

In this case, there is only one combinatorial type for the tropical plane and four possible types for the lines up to symmetry (see [MS15, Example 4.4.9]). The plane consists of six 2-dimensional cones positively spanned by all possible pairs of vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and $\mathbf{e}_0$. The combinatorial types of the tropical lines are shown in Figure 3.1. The leaves of these graphs represent the rays of the tropical line labeled 0 up to 3 corresponding to the positive hull of each of the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and $\mathbf{e}_0$.

Consider the $S_4 \rtimes \mathbb{Z}_2$-orbits of maximal cones of $\operatorname{trop} \mathcal{F}\ell_4$. If we compute the chain of tropical linear spaces corresponding to an element in each orbit, we get the configurations in Figure 3.2. Note that we do not include the labeling since up to symmetry we can get all possibilities. The point on the line is the black dot. In case the intersection of the line with the rays of the plane is the vertex of the plane then we denote this with a hollow dot. A vertex of the line is coloured in gray if it lies on a ray of the plane. For example in orbit 2, label the rays by 0 to 3 anti-clockwise starting from the top left edge. We have rays 0 and 1 in the 2-dimensional positive hull of $\mathbf{e}_0$ and $\mathbf{e}_1$. The direction of the internal edge is $\mathbf{e}_0 + \mathbf{e}_2$. The gray point is the origin and the black point has coordinates $(a, 1, 0)^T$ for $a > 1$.

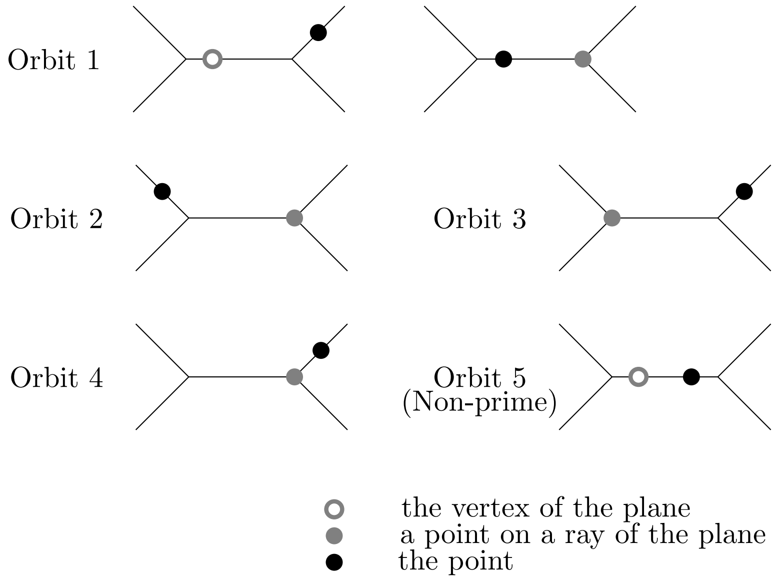

Figure 3.2: The list of all tropical configurations up to symmetry that arise in $\mathcal{F}\ell_4$. The hollow and the full gray dot denote whether that vertex of the line is the vertex of the plane or it is contained in a ray of the plane. The black dot is the position of the point on the line.

Orbits 1 and 4 in Figure 3.2 have size 24, orbits 2 and 3 have size 12 and orbit 5 has size 6. Note that orbit 5 corresponds to non-prime initial ideals. Orbit 1 contains two combinatorial types of tropical configurations and one is sent to the other by the $\mathbb{Z}_2$-action on the tropical variety. The orbits 2 and 3 differ from the fact that for each combinatorial type of line the gray dot can lie on one of the four rays of the tropical plane. These possibilities are grouped in two pairs, one is in orbit 2 and the other in orbit 3.

□

Theorem 3.3.5. *The tropical variety* $\operatorname{trop}\mathcal{F}\ell_5$ *is a* 10*-dimensional fan in* $\mathbb{R}^{30}/\mathbb{R}^4$ *with a* 4*-dimensional lineality space. It consists of* 69780 *maximal cones which are grouped in* 536 $S_5 \rtimes \mathbb{Z}_2$*-orbits. These give rise to* 531 *orbits of binomial initial ideals and among these* 180 *are prime. They correspond to* 180 *non-isomorphic toric degenerations.*

Proof. The flag variety $\mathcal{F}\ell_5$ is a 10-dimensional variety defined by 66 quadratic polynomials in the Cox ring of $\mathbb{P}^4\times\mathbb{P}^9\times\mathbb{P}^9\times\mathbb{P}^4$. These are of the form $\sum_{j\in J\setminus I}(-1)^{l_j}p_{I\cup\{j\}}p_{J\setminus\{j\}}$, where $J, I \subset \{0,\ldots,4\}$ and $l_j = \#\{k \in J : j < k\} + \#\{i \in I : i < j\}$.

The proof is similar to the proof of Theorem 3.3.3. The only difference is that the action of $S_5 \rtimes \mathbb{Z}_2$ on $\mathcal{F}\ell_5$ is crucial for the computations. In fact, without exploiting the symmetries the calculations to get the tropicalization would not terminate. Moreover, we only verify primeness of the initial ideals over $\mathbb{Q}$ using the *primdec* library [PDSL] in `Singular` [DGPS15]. We compute the polytopes associated to the normalization of the 180 toric varieties in the same way as Theorem 3.3.3, only changing the matrix of the grading. It is now given by

$$\begin{bmatrix} 1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1 \end{bmatrix}. \tag{3.3}$$

Since there are no combinatorial equivalences among the normal fans to these polytopes, we deduce that the obtained toric degenerations are pairwise non-isomorphic. More information on the non-prime initial ideals is available in Table 5.1 in the appendix. □

3.4 String cones and the tropicalized flag variety

We have seen in section 3.3 how to obtain toric degenerations from maximal prime cones of the tropicalization of the flag varieties. In this section we compare these toric degenerations to the ones arising from *string polytopes* and the FFLV polytope.

String polytopes are described by Littelmann in [Lit98], and by Berenstein and Zelevinsky in [BZ01]. FFLV stands for Feigin, Fourier, and Littelmann, who defined this polytope in [FFL11], and Vinberg who conjectured its existence in a special case. Both, the string polytopes and the FFLV polytope, can be used to obtain toric degenerations of the flag variety. The special fibre is the normal toric variety associated to the polytope ([AB04, Theorem 3.2],[Cal02],[FFL16b]).

Proposition 3.4.1. *[BLMM17, Proposition 4.8] For $\mathcal{F}\ell_4$ there are four string polytopes in $\mathbb{R}^{10}$ up to unimodular equivalence and three of them satisfy MP. For $\mathcal{F}\ell_5$ there are* 28 *string polytopes in $\mathbb{R}^{14}$ up to unimodular equivalence and* 14 *of them satisfy MP.*

We will now prove Theorem 3.1.1 by analyzing the polytopes associated to the different toric degenerations of $\mathcal{F}\ell_n$ for $n = 4, 5$. The string polytopes and the

Orbit	Combinatorially equivalent polytopes
1	String 2
2	String 1 (Gelfand-Tsetlin)
3	String 3 and FFLV
4	-

Table 3.2: Combinatorial equivalences among the polytopes obtained from prime cones in $\operatorname{trop}\mathcal{F}\ell_4$ and string polytopes resp. the FFLV polytope.

FFLV polytopes are computed in [BLMM17, Section 4]. We collect the data needed for the proof of Proposition 3.4.1 in Tables 3.2 and 5.2 (see Appendix for Table 5.2).

Proof of Theorem 3.1.1 . In order to distinguish the different toric degenerations, we compare the toric varieties associated to their special fibres. In case of the degenerations induced by the string polytopes and FFLV polytope, these toric varieties are normal. This might not be true for the degenerations found in Theorem 3.3.3 and Theorem 3.3.5. For this reason we compare the toric varieties associated to the string polytopes and the FFLV polytope with the normalization of the toric varieties associated to the cones of $\operatorname{trop}\mathcal{F}\ell_n$. These are the toric varieties whose associated polytopes are those found in the proofs of Theorem 3.3.3 and Theorem 3.3.5. Two toric degenerations are considered distinct if the normalization of the special fibres are not isomorphic toric varieties.

Two toric varieties are isomorphic, if their corresponding fans are unimodular equivalent. If the varieties are normal these are the normal fans to the associated polytopes. For this reason we first look for combinatorial equivalences between those. If they are not combinatorially equivalent then their normal fans can not be unimodular equivalent. We use `polymake` [GJ] for computations with polytopes.

From Table 3.2 one can see that for $\mathcal{F}\ell_4$ there is one toric degeneration, whose associated polytope is not combinatorially equivalent to any string polytope or the FFLV polytope for ρ. Hence, its corresponding normal toric variety is not isomorphic to any toric variety associated to these polytopes. For the toric varieties associated to the other polytopes we can not exclude isomorphism since there might be a unimodular equivalences between pairs of normal fans.

For $\mathcal{F}\ell_5$, Table 5.2 in the appendix shows that there are 168 polytopes obtained from prime cones of $\operatorname{trop}\mathcal{F}\ell_5$ that are not combinatorially equivalent to any string polytope or the FFLV polytope for ρ. $\square$

Remark 3.4.2. There are also string polytopes, which are not combinatorially equivalent to any polytope from prime cones in trop $\mathcal{F}\ell_n$ for $n = 4, 5$. See also the Table 5.2 in Appendix.

3.5 Toric degenerations from non-prime cones

As we have seen in §3.3, not all maximal cones in the tropicalization of a variety give rise to prime initial ideals and hence to toric degenerations. In fact, there may also be tropicalizations without prime cones (see Example 3.5.2). Let X be a subvariety of a toric variety Y. In this section, we give a recursive procedure (Procedure 1) to compute a new embedding X' of X in case trop X has non-prime cones. Let C be a non-prime cone. If the algorithm terminates, the new variety X' has more prime cones than trop X and at least one of them is projecting onto C. We apply this procedure to $\mathcal{F}\ell_4$ and compare the new toric degenerations with those obtained so far (see Proposition 3.5.3). The procedure terminates for $\mathcal{F}\ell_4$, but we are still investigating the conditions for which this is true in general.

Procedure 1: Computing new embeddings of the variety X in case trop X contains non-prime cones

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Input: $A = \mathbb{C}[x_0, \dots, x_n]/I$, where $\mathbb{C}[x_0, \dots, x_n]$ is the total coordinate ring of the toric variety Y and I defines the subvariety $V(I) \subset Y$, C a non-prime cone of trop $V(I)$.

Initialization:
Compute the primary decomposition of $\mathrm{in}_C(I)$;
$I(W_C)$ = unique prime toric component in the decomposition;
G = minimal generating set of $I(W_C)$.
Compute a list of binomials $L_C = \{f_1, \dots, f_s\}$ in G, which are not in $\mathrm{in}_C(I)$;
$A' = \mathbb{C}[x_0, \dots, x_n, y_1, \dots, y_s]/I'$ with $I' = I + \langle y_1 - f_1, \dots, y_s - f_s \rangle$;
$V(I')$ subvariety of Y' whose total coordinate ring is $\mathbb{C}[Y] := \mathbb{C}[x_0, \dots, x_n, y_1, \dots, y_s]$.
Compute trop $V(I')$;
for *all prime cones* $C' \in \mathrm{trop}\, V(I')$ **do**
  if $\pi(C')$ *is contained in the relative interior of* C **then**
    Output: The algebra A' and the ideal $\mathrm{in}_{C'}(I')$ of a toric degeneration of $V(I')$.
  else
    Apply the procedure again to A' and C'.

We now explain Procedure 1. Consider a toric variety Y whose total coordinate ring is $\mathbb{C}[x_0, \dots, x_n]$ with associated $\mathbb{Z}^k$-degree $\deg : \mathbb{Z}^{n+1} \to \mathbb{Z}^k$. Let X be the subvariety of Y associated to an ideal $I \subset \mathbb{C}[x_0, \dots, x_n]$, where the Krull dimension of $A = \mathbb{C}[x_0, \dots, x_n]/I$ is d. Denote by $\operatorname{trop} V(I)$ the tropicalization of X intersected with the torus of Y. Suppose there is a non-prime cone $C \subset \operatorname{trop} V(I)$ with multiplicity one. By Lemma 3.2.4, we have that $I(W_C)$ is the unique toric ideal in the primary decomposition of $\mathrm{in}_C(I)$, hence $\mathrm{in}_C(I) \subset I(W_C)$. We can compute $I(W_C)$ using the function `primaryDecomposition` in `Macaulay2`. Fix a minimal binomial generating set G of $I(W_C)$. Let $L_C = \{f_1, \dots, f_s\}$ be the set consisting of binomials in G, which are not in $\mathrm{in}_C(I)$. By Hilbert's Basis Theorem, s is a finite number. The absence of these binomials in $\mathrm{in}_C(I)$ is the reason why the initial ideal is not equal to $I(W_C)$. We introduce new variables $\{y_1, \dots, y_s\}$ and consider the algebra $A' = \mathbb{C}[x_0, \dots, x_n, y_1, \dots, y_s]/I'$, where

$$I' = I + \langle y_1 - f_1, \dots, y_s - f_s \rangle.$$

The ideal I' is a homogeneous ideal in $\mathbb{C}[x_0, \dots, x_n, y_1, \dots, y_s]$ graded by

$$(\deg(x_0), \dots, \deg(x_n), \deg(f_1), \dots, \deg(f_s)).$$

The new variety $V(I')$ is a subvariety of a toric variety Y', which has total coordinate $\mathbb{C}[Y'] := \mathbb{C}[x_0, \dots, x_n, y_1, \dots, y_s]$. For example, if $V(I)$ is a subvariety of a projective space then $V(I')$ is contained in a weighted projective space.

Since the algebras A and A' are isomorphic as graded algebras, the varieties $V(I)$ and $V(I')$ are isomorphic. We have a monomial map

$$\pi : \mathbb{C}[x_0, \dots, x_n]/I \to \mathbb{C}[x_0, \dots, x_n, y_1, \dots, y_s]/I'$$

inducing a surjective map $\operatorname{trop}(\pi) : \operatorname{trop} V(I') \to \operatorname{trop} V(I)$ (see [MS15, Corollary 3.2.13]). The map $\operatorname{trop}(\pi)$ is the projection onto the first n coordinates. Suppose there exists a prime cone $C' \subset \operatorname{trop} V(I')$, whose projection has a non-empty intersection with the relative interior of C. Then by construction we have $\mathrm{in}_C(I) \subset \mathrm{in}_{C'}(I') \cap \mathbb{C}[x_0, \dots, x_n]$ and the procedure terminates. In this way we obtain a new initial ideal $\mathrm{in}_{C'}(I')$ which is toric and hence gives a new toric degeneration of the variety $V(I') \cong V(I)$. If only non-prime cones are projecting to C then run this procedure again with A' and C', where the latter is a maximal cone of $\operatorname{trop} V(I')$, which projects to C. We can then repeat the procedure starting from a different non-prime cone.

The function to apply Procedure 1 is `findNewToricDegenerations` and it is part of the package `ToricDegenerations`. This will compute only one re-embedding for each non-prime cone. It is possible to use `mapMaximalCones` to obtain the image of $\operatorname{trop} V(I')$ under the map π.

Remark 3.5.1. If f_i is a polynomial in $\mathbb{K}[x_0, x_1, \ldots, x_n]$ with the standard grading and $\deg(f_i) > 1$, then we need to homogenize the ideal I' before computing the tropicalization with `Gfan`. This is done by adding a new variable h. The homogenization of I' with respect to h is denoted by $I'_{proj} \subseteq \mathbb{K}[x_0, \ldots, x_n, y_1, \ldots, y_s, h]$. Then by [MS15, Proposition 2.6.1] for every $\mathbf{w}$ in $\mathbb{R}^{n+s+2}$ the ideal $\operatorname{in}_{\mathbf{w}}(I')$ is obtained from $\operatorname{in}_{(\mathbf{w},0)}(I'_{proj})$ by setting $h = 1$.

If the cone C is prime, we can construct a valuation val_C on $\mathbb{K}[x_0, \ldots, x_n]/I$ in the following way. Consider the matrix W_C in Equation (3.1). For monomials $m_i = c\ \mathbf{x}^{\alpha_i} \in \mathbb{K}[x_0, \ldots, x_n]$ define

$$\operatorname{val}(m_i) = W_C \cdot \alpha_i \quad \text{and} \quad \operatorname{val}(\sum_i m_i) = \min_i\{\operatorname{val}(m_i)\}, \tag{3.4}$$

where the minimum on the right side is taken with respect to the lexicographic order on $(\mathbb{Z}^d, +)$. This is a valuation on $\mathbb{K}[x_0, \ldots, x_n]$ of rank equal to the Krull dimension of A for every cone C. Composing val with the quotient morphism $p : \mathbb{K}[x_0, \ldots, x_n] \to \mathbb{K}[x_0, \ldots, x_n]/I$ we obtain a map val_C, which is a valuation if and only if the cone C is prime. Moreover, in [KM16] Kaveh and Manon prove that a cone C in $\operatorname{trop} V(I)$ is prime if and only if $A = \mathbb{K}[x_0, \ldots, x_n]/I$ has a finite *Khovanskii basis* for the valuation val_C constructed from the cone C. Recall that a Khovanskii basis for an algebra A with valuation val_C is a subset B of A such that $\operatorname{val}_C(B)$ generates the value semigroup $S(A, \operatorname{val}_C) = \{\operatorname{val}_C(f) : f \in A \setminus \{0\}\}$.

Procedure 1 can be interpreted as finding an extension $\operatorname{val}_{C'}$ of val_C so that A' has finite Khovanskii basis with respect to $\operatorname{val}_{C'}$. The Khovanskii basis is given by the images of $x_0, \ldots, x_n, y_1, \ldots, y_s$ in A'. We illustrate the procedure in the following example.

Example 3.5.2. Consider the algebra $A = \mathbb{C}[x, y, z]/\langle xy + xz + yz\rangle$. The tropicalization of $V(\langle xy+xz+yz\rangle) \subset \mathbb{P}^2$ has three maximal cones. The corresponding initial ideals are $\langle xz + yz\rangle, \langle xy + yz\rangle$ and $\langle xy + xz\rangle$, none of which is prime. Hence they do not give rise to toric degenerations. The matrices associated to each cone are

$$W_{C_1} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}, \quad W_{C_2} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad W_{C_3} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

We now apply Procedure 1 to the cone C_1. The initial ideal associated to C_1 is generated by $xz + yz$. In this case $\operatorname{in}_{C_1}(I) = \langle z\rangle \cdot \langle x + y\rangle$ hence for the missing binomial $x+y$ we adjoin a new variable u to $\mathbb{C}[x,y,z]$ and the new relation $u-x-y$ to I. We have

$$I' = \langle xy + xz + yz, u - x - y\rangle \text{ and } A' = \mathbb{C}[x,y,z,u]/I'$$

with $V(I')$ a subvariety of $\mathbb{P}^3$. After computing the tropicalization of $V(I')$ we see that there exists a prime cone C' such that $\pi(C') = C$. The matrix $W_{C'}$ associated to the cone C' is

$$W' = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

The initial ideal $\mathrm{in}_{C'}(I')$ gives a toric degeneration of $V(I')$. The image of the set $\{x,y,z,u\}$ in A' is a Khovanskii basis for $S(A', \mathrm{val}_{C'})$. Repeating this process for the cones C_2 and C_3 of $\mathrm{trop}\, V(xy + xz + yz)$, we get prime cones C_2' and C_3' whose projections are C_2 and C_3 respectively. Hence there is a valuation with finite Khovanskii basis and a corresponding toric degeneration for every maximal cone.

We now apply Procedure 1 to $\mathrm{trop}\,\mathcal{F}\ell_4$.

Proposition 3.5.3. *Each of the non-prime cones of* $\mathrm{trop}\,\mathcal{F}\ell_4$ *gives rise to three toric degenerations, which are not isomorphic to any degeneration coming from the prime cones of* $\mathrm{trop}\,\mathcal{F}\ell_4$*. Moreover, two of the three new polytopes are combinatorially equivalent to the previously missing string polytopes for ρ in the class String 4.*

Proof. By Theorem 3.3.3 we know that $\mathrm{trop}\,\mathcal{F}\ell_4$ has six non-prime cones forming one $S_4 \rtimes \mathbb{Z}_2$-orbit. Hence, we apply Procedure 1 to only one non-prime cone. The result for the other non-prime cones will be the same up to symmetry. In particular, the obtained toric degenerations from one cone will be isomorphic to those coming from another cone. We describe the results for the maximal cone C associated to the initial ideal $\mathrm{in}_C(I_4)$ defined by the following binomials:

$$\begin{array}{llll} p_3p_{012} - p_2p_{013}, & p_{13}p_{023} - p_{03}p_{123}, & p_{12}p_{023} - p_{02}p_{123}, & p_1p_{03} - p_0p_{13}, \\ p_1p_{02} - p_0p_{12}, & p_{13}p_{012} - p_{12}p_{013}, & p_{03}p_{012} - p_{02}p_{013}, & p_3p_{12} - p_2p_{13} \\ & p_3p_{02} - p_2p_{03}, \text{ and} & p_{03}p_{12} - p_{02}p_{13}. & \end{array}$$

We define the ideal $I' = I_4 + \langle w - p_1p_{023} + p_0p_{123}\rangle$. The grading on the variables $p_1, \ldots, p_{123}$ and w is given by the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

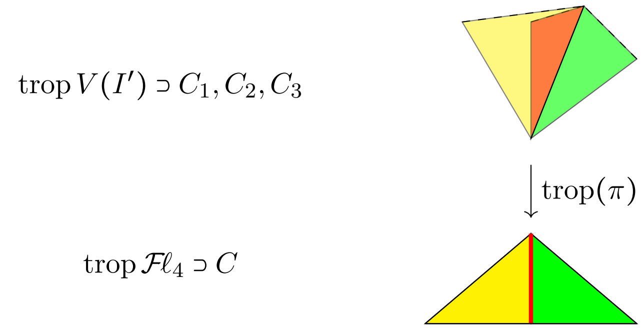

Figure 3.3: The three triangles above represent the three cones in $\operatorname{trop} V(I')$ which project down to the non-prime cone C in $\operatorname{trop} \mathcal{F}\ell_4$.

It extends the grading on the variables $p_0, \ldots, p_{123}$ given by the matrix D in (3.2). The tropical variety $\operatorname{trop} V(I')$ has 105 maximal cones, 99 of which are prime. Among them we can find three maximal prime cones, which are mapped to C by $\operatorname{trop}(\pi)$ (see Figure 3.3). We compute the polytopes associated to the normalization of these three toric degenerations by applying the same methods as in Theorem 3.3.3. Using `polymake` we check that two of them are combinatorially equivalent to the string polytopes for ρ in the class String 4. Moreover, none of them is combinatorially equivalent to any polytope coming from prime cones of $\operatorname{trop} \mathcal{F}\ell_4$, hence they define different toric degenerations. $\square$

Remark 3.5.4. Procedure 1 could be applied also to $\mathcal{F}\ell_5$ but we have not been able to do so. In fact, the tropicalization for $\operatorname{trop} V(I_5')$ did not terminate since the computation can not be simplified by symmetries.

Chapter 4

Tropical Fano schemes

4.1 Introduction

The classical Fano scheme of a projective variety $X \subset \mathbb{P}^n$ is the fine moduli space parametrising linear spaces contained in X. It is denoted by $\mathrm{F}_d(X)$, where d is the dimension of the linear spaces. It is a subvariety of the Grassmannian $\mathrm{Gr}(d+1, n+1)$. Fano schemes have been intensively studied because of their geometric properties. Gino Fano [Fan04] first introduced them. He mostly considered the case of hypersurfaces. Then in the $70s$ these schemes have been used to prove results on the irrationality of cubic threefolds ([CG72],[Mur72])). Recently there has been new interests for Fano schemes not only in algebraic geometry ([IZ17], [Ilt16], [IS17], [CI15]) but also in machine learning [LK14] and geometric complexity theory [MS01].

In this chapter we study a tropical version of the Fano scheme. We investigate the structure of this tropical object and the relations with the classical $F_d(X)$.

The first way of obtaining a tropical version of $\mathrm{F}_d(X)$ is to consider its tropicalization inside $\mathrm{trop}^{\mathrm{ext}}\,\mathrm{Gr}(d+1, n+1)$. The points of $\mathrm{trop}^{\mathrm{ext}}\,\mathrm{F}_d(X)$ are in correspondence with the tropicalization of the classical linear spaces contained in X. However it is not true in general that a tropicalized linear space that lies in $\mathrm{trop}^{\mathrm{ext}}\,X$ is the tropicalization of a classical linear space in X. A famous example of this is in [Vig10] where Vigeland proves that there are smooth surfaces in $\mathbb{P}^3$ of degree three whose tropicalization contains infinitely many lines. Since there are only finitely many tropical lines in the classical surfaces we deduce that these infinite tropical lines are not the tropicalization of the classical lines.

This leads us to define the *tropical Fano scheme* $\mathrm{F}_d(\operatorname{trop}^{\mathrm{ext}} X)$ to be the set of tropicalized linear spaces Γ of dimension d contained in $\operatorname{trop}^{\mathrm{ext}} X$. In particular we will focus on $d = 1$. We take the first steps in studying the properties of this object that can be used to study the classical Fano scheme. The theorem below allows us to consider the dimension of $\mathrm{F}_1(\operatorname{trop}^{\mathrm{ext}} X)$ that gives a bound for the dimension of $F_1(X)$.

Theorem 9. *Let X be a projective variety in $\mathbb{P}^n$. The tropical Fano scheme $\mathrm{F}_1(\operatorname{trop}^{\mathrm{ext}} X)$ is a polyhedral complex whose support is contained in* $\operatorname{trop}^{\mathrm{ext}} \mathrm{Gr}(2, n+1)$.

The other main focus of the paper is to investigate the difference between $\operatorname{trop}^{\mathrm{ext}} \mathrm{F}_1(X)$ and $\mathrm{F}_1(\operatorname{trop}^{\mathrm{ext}} X)$. In fact we immediately observe that

$$\operatorname{trop}^{\mathrm{ext}} \mathrm{F}_1(X) \subset \mathrm{F}_1(\operatorname{trop}^{\mathrm{ext}} X) \tag{4.1}$$

and a natural question arises:

Question 10. *For which varieties X do we have* $\operatorname{trop}^{\mathrm{ext}} \mathrm{F}_1(X) = \mathrm{F}_1(\operatorname{trop}^{\mathrm{ext}} X)$*?*

We start by looking at the simplest algebraic varieties which are linear subspaces of $\mathbb{P}^n$. Then we analyse the case of toric varieties embedded in $\mathbb{P}^n$ via monomial maps. These are two examples where the tropicalization is well understood and can be easily deduced from the properties of the varieties. For a linear space L the tropicalization can be computed from the matroid associated to L. The tropicalization of a toric variety X is instead the image of a linear map defined by the lattice points in $\mathcal{A} \subset \mathbb{Z}^m$ associated to the embedding of X.

Theorem 11.

1. *If L is a generic 2–dimensional plane in $\mathbb{P}^5$ then* $\operatorname{trop}^{\mathrm{ext}} \mathrm{F}_1(L) \subsetneq \mathrm{F}_1(\operatorname{trop}^{\mathrm{ext}} L)$.
2. *If X is a toric variety in $\mathbb{P}^n$ then* $\mathrm{F}_1(\operatorname{trop}^{\mathrm{ext}} X) = \operatorname{trop}^{\mathrm{ext}} \mathrm{F}_1(X)$.

The chapter is organised in the following way. In section 4.2 we study the structure of $\mathrm{F}_1(\operatorname{trop}^{\mathrm{ext}} X)$. We give a necessary and sufficient condition for a tropical line to be contained in a tropical variety and then we prove in Theorem 4.2.2 that $\mathrm{F}_1(\operatorname{trop}^{\mathrm{ext}} X)$ is a polyhedral complex. In section 4.3 we study the case of linear spaces. We prove the first part of Theorem 11 in Theorem 4.3.3 and then we use it to prove the strict containment in (4.1) for a generic hypersurface X (Proposition 4.3.7). Finally in section 4.4 we prove the second part of Theorem 11 (Theorem 4.4.2).

4.2 Definitions and properties of $\mathrm{F}_1(\operatorname{trop} X)$

In this section we define the tropical Fano Scheme $\mathrm{F}_d(\operatorname{trop}^{\mathrm{ext}} X)$ of the tropicalization of a projective variety $X \subset \mathbb{P}^n$. We focus our attention on the case of $\mathrm{F}_1(\operatorname{trop}^{\mathrm{ext}} X)$ and we define a polyhedral structure on it.

Let $\mathbb{K}$ be a field with a surjective valuation $\operatorname{val}: \mathbb{K}^* \to \mathbb{R}$ (cf. Remark 4.2.19) and T^m be the torus $(\mathbb{K}^*)^{m+1}/\mathbb{K}^*$ contained in $\mathbb{P}^m$.
For any projective variety $Y \subset \mathbb{P}^m$ the extended tropicalization $\operatorname{trop}^{\mathrm{ext}} Y$ was defined in section 2.5. The tropicalization $\operatorname{trop} \mathbb{P}^m$ is given by the union of $\mathcal{O} := \operatorname{trop} O = \mathbb{R}^{m'+1}/\mathbb{R}\mathbf{1}$ for $m' \leq m$ where $O \cong T^{m'}$ is a T^m-orbit of $\mathbb{P}^m$ that is the locus of points in $\mathbb{P}^m$ where some of the coordinates vanish. In the following we refer to $\mathcal{O}$ as an orbit of $\operatorname{trop} \mathbb{P}^n$. In the figures we identify $\mathbb{R}^{m+1}/\mathbb{R}\mathbf{1}$ with $\mathbb{R}^m$ with the isomorphism described in (2.5). Recall that $\operatorname{trop}^{\mathrm{ext}}(Y \cap \mathcal{O}) = \operatorname{trop}(Y \cap O)$ where O is the unique orbit of $\mathbb{P}^n$ such that $\operatorname{trop} O = \mathcal{O}$.

Let $\mathbb{G}(d,n) := \mathrm{Gr}(d+1, n+1)$ be the Grassmannian parametrising d-dimensional vector spaces in $\mathbb{P}^n$. We consider it embedded via the Plücker map into $\mathbb{P}^{\binom{n+1}{d+1}-1}$ and we have that its extended tropicalization $\operatorname{trop}^{\mathrm{ext}} \mathbb{G}(d,n) \subset \operatorname{trop} \mathbb{P}^{\binom{n+1}{d+1}-1}$ parametrises tropicalized linear spaces of dimension d in $\operatorname{trop} \mathbb{P}^n$ ([SS04, Theorem 3.8], [MS15, Theorem 4.3.17 and Remark 4.4.2],[CC]). Hence it is possible to associate to each point p of $\operatorname{trop}^{\mathrm{ext}} \mathbb{G}(d,n)$ a unique tropical line that we denote by Γ_p.

Definition 4.2.1. The *tropical Fano scheme* is the set $\mathrm{F}_d(\operatorname{trop}^{\mathrm{ext}} X) \subset \operatorname{trop}^{\mathrm{ext}} \mathbb{G}(d,n)$ defined by

$$\mathrm{F}_d(\operatorname{trop}^{\mathrm{ext}} X) := \{p \in \operatorname{trop}^{\mathrm{ext}} \mathbb{G}(d,n) : \Gamma_p \subset \operatorname{trop}^{\mathrm{ext}} X\}.$$

As it is defined $\mathrm{F}_d(\operatorname{trop}^{\mathrm{ext}} X)$ is a set contained in the support of $\operatorname{trop}^{\mathrm{ext}} \mathbb{G}(d,n)$. In the rest of the section we focus our attention on $\mathrm{F}_1(\operatorname{trop}^{\mathrm{ext}} X)$ and prove the following result:

Theorem 4.2.2. *Let X be a projective variety in $\mathbb{P}^n$ and $\mathcal{O}$ be an orbit of* $\operatorname{trop} \mathbb{P}^{\binom{n+1}{2}-1}$. *Then $\mathrm{F}_1(\operatorname{trop}^{\mathrm{ext}} X) \cap \mathcal{O}$ is a polyhedral complex whose support is contained in $\operatorname{trop}^{\mathrm{ext}} \mathbb{G}(1,n) \cap \mathcal{O}$.*

In order to prove Theorem 4.2.2 we first stratify $\mathrm{F}_1(\operatorname{trop}^{\mathrm{ext}} X) \cap \mathcal{O}$ with sets denoted by C_S^T. Then in Propositions 4.2.14 and 4.2.15 we show that the closure of any such set is a polyhedron contained in $\mathrm{F}_1(\operatorname{trop}^{\mathrm{ext}} X) \cap \mathcal{O}$. Finally in the proof of Theorem 4.2.2 we show that the collection of these polyhedra form a polyhedral complex.

The polyhedral structure that we define on $\mathrm{F}_1(\operatorname{trop} X) \cap \mathcal{O}$ is modelled on the polyhedral structure of $\operatorname{trop}^{\text{ext}} \mathbb{G}(1,n) \cap \mathcal{O}$ induced by the identification of $\operatorname{trop}^{\text{ext}} \mathbb{G}(1,n) \cap \mathcal{O}$ with the space of metrics on phylogenetic trees. The case of $\operatorname{trop} \mathbb{G}(1,n) = \operatorname{trop} \mathbb{G}(1,n) \cap \operatorname{trop} T^{\binom{n+1}{2}-1}$ was studied by Speyer and Sturmfels in [SS04]. It has been recently generalised to all the other orbits $\mathcal{O}$ of $\operatorname{trop} \mathbb{P}^n$ by Cueto and Corey in [CC].

We describe the polyhedral structure for $\operatorname{trop} \mathbb{G}(1,n) = \operatorname{trop}^{\text{ext}} \mathbb{G}(1,n) \cap \operatorname{trop} T^{\binom{n+1}{2}-1}$ and then we explain how to generalise it to $\operatorname{trop}^{\text{ext}} \mathbb{G}(1,n) \cap \mathcal{O}$ as in [CC].

Firstly we recall that $\mathbb{G}(1,n) \cap O \cong \mathrm{Gr}_M$ with M the rank 2 matroid on $\{0,1,\dots,n\}$ whose bases are $\{i,j\}$ such that $p_{ij} \neq 0$ in $\mathbb{G}(1,n) \cap O$ (see [MS15, Section 4.4.1] for a definition of Gr_M). The isomorphism $\phi : \mathbb{G}(1,n) \cap O \to \mathrm{Gr}_M$ is the projection onto to the p_{ij}'s that do not vanish. The corresponding map of tropical varieties is

$$\operatorname{trop} \phi : \operatorname{trop}(\mathbb{G}(1,n) \cap O) \to \operatorname{trop} \mathrm{Gr}_M$$

that is the projection onto the p_{ij}'s that are not infinity. Here we abuse notation using p_{ij} for the coordinates of $\mathbb{R}^{m+1}/\mathbb{R}\mathbf{1} = \operatorname{trop} O = \mathcal{O}$. From now on we identify $\mathbb{G}(1,n) \cap O$ with Gr_M and $\operatorname{trop}(\mathbb{G}(1,n) \cap O)$ with $\operatorname{trop} \mathrm{Gr}_M$. The intersection $\mathbb{G}(1,n) \cap O = \mathrm{Gr}_M$ parametrises lines contained in the orbit $O'_M = \{[x_0 : \dots : x_n] \in \mathbb{P}^n : x_i = 0$ if and only if $\{i\}$ is a circuit (loop) of $M\}$. The tropicalization $\operatorname{trop}(\mathbb{G}(1,n) \cap O)$ parametrises tropical lines contained in $\operatorname{trop} O'_M = \mathcal{O}'_M = \mathbb{R}^{m+1}/\mathbb{R}\mathbf{1}$ where $m = n - l$ with l number of loops in M. For $\mathbb{G}(1,n) \cap T^{\binom{n+1}{2}-1}$ the matroid M is the uniform matroid $U_{2,n+1}$, the orbit $O'_M = \{[x_0 : \dots : x_n] \in \mathbb{P}^n : x_i \neq 0 \ \forall i\} = T^n$ and $\mathcal{O}'_M = \mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$.

Remark 4.2.3. Let Γ be a tropical line in $\operatorname{trop} \mathbb{P}^n$ and assume $\mathcal{O}'$ is the unique orbit of $\operatorname{trop} \mathbb{P}^n$ such that $\dim(\Gamma \cap \mathcal{O}') = 1$. Since $\Gamma = \overline{\Gamma \cap \mathcal{O}'}$ in $\operatorname{trop} \mathbb{P}^n$ it is enough to study $\Gamma \cap \mathcal{O}'$. In order to make the notation simpler we will write Γ for $\Gamma \cap \mathcal{O}'$ unless we state otherwise.

The tropicalization $\operatorname{trop} \mathrm{Gr}_{U_{2,n+1}}$ is equal to the space of phylogenetic trees on $n+1$ leaves. This is the space of metrics on trees with leaves labelled by integers $i \in \{0,1,\dots,n\}$ ([SS04, Theorem 3.4]). This identification induces a polyhedral

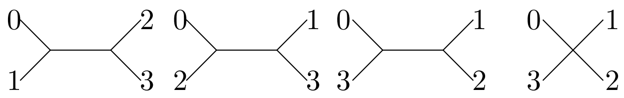

Figure 4.1: Trees associated to the cones of $\mathrm{trop}(\mathbb{G}(1,3)\cap T^5)$.

structure on $\mathrm{trop}\,\mathbb{G}(1,n)\cap T^{\binom{n+1}{2}-1}$ given by the union of cones C_T where each T is a different labelled tree and each point in the relative interior C_T° of C_T corresponds to a metric on the tree T. The tree $T_{\Gamma_p} := T$ gives information on the finest polyhedral structure Σ on Γ_p for any $p \in C_T$ ([SS04, Corollary 6.1]). In fact there is a bijection between 1-dimensional cells (*resp.* vertices) of Σ and edges (*resp.* non-leaf vertices) of T that is compatible with adjacency between the 1-dimensional cells. We will denote by $\overline{V}$ the vertex of T corresponding to a vertex V of Γ_p. Let i be the label on a leaf $\overline{V}$ and let $\overline{V}'$ be the vertex of T_Γ adjacent to $\overline{V}$. Then the corresponding edge of Γ_p is $\mathrm{pos}(\mathbf{e}_{i+1}) + V' := \{\lambda \mathbf{e}_{i+1} + V' : \lambda > 0\} \subset \mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ where $\overline{V}'$ is the vertex of T_Γ adjacent to $\overline{V}$ that is $e = VV'$. Applying the balancing condition it is possible to deduce the generator of the linear span of all the other edges of Σ. We refer to this generator as the direction of the edge. This is unique up to scaling.

Example 4.2.4. Consider $\mathrm{Gr}(2,4) = \mathbb{G}(1,3) \subset \mathbb{P}^5$ and the tropical variety $\mathrm{trop}(\mathbb{G}(1,3)\cap T^5) \subset \mathbb{R}^6/\mathbb{R}\mathbf{1}$. This is a 4-dimensional fan with three cones that intersect in a 3-dimensional lineality space L. The three cones are labelled by the first three trees in Figure 4.1, while L is labelled by the last one.

Consider now $\mathrm{trop}\,\mathrm{Gr}_M$ with $M \neq U_{2,n+1}$. In [CC] the authors show that there is an isomorphism

$$\psi : \mathrm{Gr}_M \to \mathrm{Gr}_{U_{2,n'+1}} \times \prod_{j\in J} T^j \tag{4.2}$$

with $n' < n$ and J a finite subset of $\mathbb{N}$. Moreover the map ψ is monomial and it induces a linear isomorphism (cf. [MS15, Section 2.6])

$$\mathrm{trop}(\psi) : \mathrm{trop}\,\mathrm{Gr}_M \to \mathrm{trop}(\mathrm{Gr}_{U_{2,n'+1}} \times \prod_{j\in J} T^j) \tag{4.3}$$

hence it is possible to consider on $\mathrm{trop}\,\mathrm{Gr}_M$ the same polyhedral structure of $\mathrm{trop}(\mathrm{Gr}_{U_{2,n'+1}} \times \prod_{j\in J}(\mathbb{K}^*)^j)$) ([CC]). Each cone of Gr_M is identified with a cone of $\mathrm{trop}\,\mathrm{Gr}_{U_{2,n'+1}}$ plus a linear space that comes from $\mathrm{trop}\prod_{j\in J}(\mathbb{K}^*)^j = \mathbb{R}^{\sum_{j\in J} j}/\mathbb{R}\mathbf{1}$. This implies that $\mathrm{trop}\,\mathrm{Gr}_M$ can be identified with the space of phylogenetic trees with $n'+1$ leaves but in this case these will be labelled by subsets of $\{0,\dots,m\}$ where $m = n-l$ and l is the number of loops in M. As before given $p \in C_T$ we have a bijection between $T_{\Gamma_p} := T$ and the finest polyhedral structure on Γ_p. The only change will be that if a leaf is labelled by I then the corresponding ray of Γ_p will be $V + \mathrm{pos}(\sum_{i\in I} \mathbf{e}_{i+1}) \subset \mathcal{O}'_M = \mathbb{R}^{m+1}/\mathbb{R}\mathbf{1}$.

Finally we recall that for every M each point p in $\operatorname{trop}\operatorname{Gr}_M$ induces a regular subdivision Δ_p of the matroid polytope

$$P_M = \mathrm{conv}(\{\mathbf{e}_{i,j} : \{i,j\} \text{ is a basis of } M \subset \mathbb{R}^{n+1}\}).$$

If M has no loops then Γ_p is a subcomplex Σ' of the dual complex to Δ_p as we have seen in section 2.4. Moreover for $M = U_{2,m}$ then any point $p \in C_T^\circ$ induces the same subdivision of P_M ([GM10, Proposition 5.5]) hence there is a correspondence between the polyhedral structure on Γ_p induced by the dual complex Σ' and T for any $p \in C_T$. This will be crucial in Lemma 4.2.17.

In the cases in which M has some loops then consider $U_{2,n'+1}$ such that $\operatorname{Gr}_M \cong \operatorname{Gr}_{U_{2,n'+1}} \times \prod_{j\in J} T^j$ via the map ψ. Then from the definition of ψ we have that $\operatorname{trop}(\psi)$ induces a map l_ψ from the tropical lines associated to points of $\operatorname{trop}\operatorname{Gr}_{U_{2,n'+1}}$ to the ones associated to points in $\operatorname{trop}(\operatorname{Gr}_M)$. Hence it is possible to compute the tropical line associated to $\operatorname{trop}(\psi)(p)$ and then Γ_p will be equal to $l_\psi^{-1}(\Gamma_{\operatorname{trop}(\psi)(p)})$.

Remark 4.2.5. There are cases in which Γ_p is a classical line. Then T_{Γ_p} is a tree with only one edge whose end points are both leaves.

Example 4.2.6. Consider $\mathbb{G}(1,4)$ and the orbit O of $\mathbb{P}^9$

$$O = \{p_{ij} = 0 \text{ if and only if } \{ij\} \in \{\{01\},\{02\},\{03\},\{04\},\{34\}\}\}.$$

The intersection $\mathbb{G}(1,4)\cap O$ is identified with Gr_M with M the matroid on $\{0,1,2,3,4\}$ whose bases are $\{1,2\},\{1,3\},\{1,4\},\{2,3\}$, $\{2,4\}$. The loop $\{0\}$ is a circuit of M. We have an isomorphism ψ_1 between Gr_M and $\operatorname{Gr}_{U_{2,4}} \cap \{p_{23} = 0\}$. Let p_{ij} and p'_{ij} be the Plücker coordinates of Gr_M and $\operatorname{Gr}_{U_{2,4}}$ respectively. The isomorphism sends p_{ij} to $p'_{(i-1)(j-1)}$. Now we have that $\operatorname{Gr}_{U_{2,4}} \cap \{p_{23} = 0\}$ is isomorphic to $\operatorname{Gr}_{U_{2,3}} \times T^1$ and the isomorphism ψ_2 is $\pi \times f$ where π is the projection onto p_{01}, p_{02}, p_{12} and f sends a point $p \in \operatorname{Gr}_{U_{2,3}}$ to p_{03}/p_{02}.

As in the case of $\operatorname{trop}^{\mathrm{ext}} \mathbb{G}(1,n)$ each stratum of $\mathrm{F}_1(\operatorname{trop}^{\mathrm{ext}} X)$ will be labelled by particular graphs. These will record not only the tree associated to the tropical line but also the way it is contained in $\operatorname{trop}^{\mathrm{ext}} X$.

Remark 4.2.7. Let $\mathcal{O}'$ be the unique orbit of $\operatorname{trop}\mathbb{P}^n$ such that $\dim(\Gamma \cap \mathcal{O}') = 1$. We have that $\Gamma \subset \operatorname{trop}^{\text{ext}} X$ if and only if $\Gamma \cap \mathcal{O}' \subset \operatorname{trop}^{\text{ext}} X \cap \mathcal{O}'$. In fact $\Gamma = \overline{\Gamma \cap \mathcal{O}'} \subset \overline{\operatorname{trop} X \cap \mathcal{O}'} \subset \operatorname{trop} X$. For this reason we simplify the notation and we write $\Gamma \subset \operatorname{trop} X$ instead of $\Gamma \cap \mathcal{O}' \subset \operatorname{trop}^{\text{ext}} X \cap \mathcal{O}'$.

Let Σ' be the polyhedral structure given by the common refinement of the structure on $\operatorname{trop} X$ with Σ, where Σ is the coarsest polyhedral structure on Γ. This means that if σ is a cell of Σ and $\sigma = (\sigma \cap \tau_1) \cup \ldots \cup (\sigma \cap \tau_n)$, with τ_i cell of $\operatorname{trop} X$, then in Σ' the cell σ is replaced by the $\sigma \cap \tau_i$'s.

We can associate to Σ' a unique labelled graph $\mathcal{S}$ whose vertices and edges are in correspondence with the cells of Σ'. We say that $\mathcal{S}$ is the labelled subdivision of T_Γ *induced* by $\Gamma \subset \operatorname{trop} X$.

With the same convention as before we denote by $\overline{V}$ a vertex of $\mathcal{S}$, and by V the corresponding vertex of Γ. The tree $\mathcal{S}$ is a subdivision of the tree T_Γ with labels on the vertices. It is obtained by replacing the edges of T_Γ with paths corresponding to cells of Σ that are subdivided in the new polyhedral structure Σ'. Every vertex is labelled with the cell of $\operatorname{trop} X$ that contains the corresponding vertex of Γ in its relative interior. Let $\overline{V}$ be a leaf labelled by I then the second label of this vertex is the cell that contains $W + \operatorname{pos}(\sum_{i \in I} \mathbf{e}_i)$ in its relative interior, where $\overline{W}$ is the vertex adjacent to $\overline{V}$. Thus to each tropical line $\Gamma \subset \operatorname{trop} X$ is associated a unique labelled subdivision $\mathcal{S}$ of T_Γ where all vertices are labelled with cells of $\operatorname{trop} X$. Moreover the labelling satisfies the following properties:

Property 4.2.8. If $(\overline{W_1}, \sigma_1)$ and $(\overline{W_2}, \sigma_2)$ are the end points of an edge of $\mathcal{S}$ then there exists a cell $\sigma \subset \operatorname{trop} X$ such that σ_1 and σ_2 are faces of σ.

Property 4.2.9. If $(\overline{V_1}, \sigma_1), \ldots, (\overline{V_1}, \sigma_s)$ are vertices of a path in $\mathcal{S}$ that replaces an edge of T_Γ then $\sigma_1, \ldots, \sigma_s$ are s distinct cells of $\operatorname{trop} X$.

Example 4.2.10. Let $\mathbb{K}$ be the field of generalised Puiseux series $\mathbb{C}((\mathbb{R}))$ with the natural valuation val associated to it (see [MS15, Example 2.17]). Consider the plane $L = V(x_0 + x_1 + x_2 + x_3)$ in $\mathbb{P}^3$ and let Γ be a tropical line contained in $\operatorname{trop}^{\text{ext}} L$ as in Figure 4.2. We denote by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ the standard basis of $\mathbb{R}^3$ and by $\mathbf{e}_0$ the vector $-\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3$. Denote by $\tau_{0,3}$ the cone spanned by $\mathbf{e}_0$ and $\mathbf{e}_3$, by $\mathbf{0}$ the point $(0,0,0)$ and by $\tau_{1,2}$ the cone spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$. In Figure 4.3 we show the tree T_Γ and the labelled graph $\mathcal{S}$ associated to the tropical line Γ.

Labelled subdivisions of T_Γ can be used to check whether Γ is contained in $\operatorname{trop} X$. This is done by applying the following procedure.

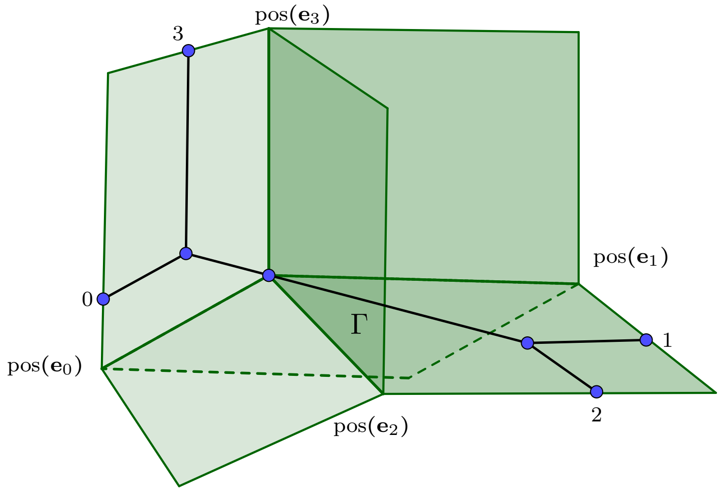

Figure 4.2: The tropical plane $\operatorname{trop} L$ and a tropical line Γ contained in it as in Example 4.2.10.

Procedure 4.2.11. [Procedure to check whether Γ is contained in $\operatorname{trop} X$ and the labelled subdivision of T_Γ induced by $\Gamma \subset \operatorname{trop} X$ is $\mathcal{S}$]
Consider a tropical line Γ and a labelled subdivision $\mathcal{S}$ of T_Γ that satisfies Properties 4.2.8 and 4.2.9.

Step 1: For every leaf $\overline{V}$ of $\mathcal{S}$ labelled by (σ, I) check whether $\mathbf{e}_I = \sum_{i \in I} \mathbf{e}_i$ is in the relative interior of the recession cone of σ. If this holds for every leaf then continue.

Step 2: For every $(\overline{V}, \sigma)$ vertex of $\mathcal{S}$ that is also a vertex of T_Γ check whether V is contained in the relative interior of σ. If this holds for every vertex then continue.

Step 3: Let $e = \overline{VV'}$ be an edge of T_Γ that is replaced by a path in $\mathcal{S}$. Let $\mathbf{v}$ be the direction of the corresponding cell of Γ and $(\overline{V}, \sigma), (\overline{V_1}, \sigma_1), \dots, (\overline{V_s}, \sigma_s), (\overline{V'}, \sigma')$ be the sequence of vertices in the path.
Intersect the line passing through V and direction $\mathbf{v}$ with the affine span of σ_1. If there exists a unique point of intersection and it is contained in the relative interior σ_1 then this is the point V_1 in Γ. Continue to apply the same procedure to V_1 to find V_2. Repeat the same construction with any V_i for $i = 2, \dots, s-1$ as long as V_{i-1} exists.

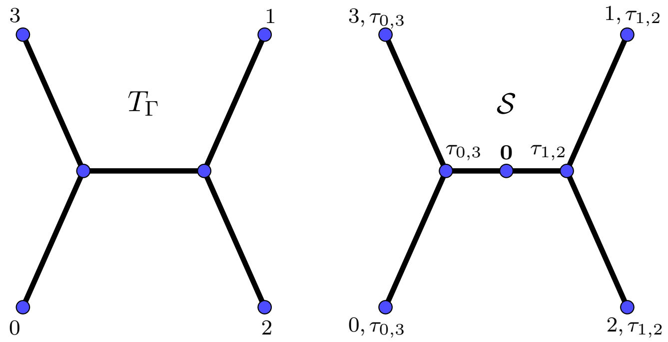

Figure 4.3: The tree T_Γ and its subdivision induced by the containment of the tropical line Γ inside the standard tropical plane in Figure 4.2.

If it is possible to compute $V_1, \dots, V_s$ then the segment $VV_s \subset e$ is contained in $\operatorname{trop} X$. In fact VV_1 is contained in the smallest cell of $\operatorname{trop} X$ that has σ and σ_1 as faces. Then for every i we have that V_i and V_{i+1} are contained respectively in σ_i° and σ_{i+1}° hence the segment $\overline{V_i V_{i+1}}$ is contained in $\sigma_{i,i+1}$ that is a cell of $\operatorname{trop} X$ that has σ_i and σ_{i+1} as faces. Finally if V_s is contained in the line passing through V_{s-1} and with direction $\mathbf{v}$ the cell corresponding to e is in $\operatorname{trop} X$.
In the case $(\overline{V'}, \sigma')$ is a leaf labelled by I then we have instead to check whether $V_s + \operatorname{pos}(\sum_{i \in I} \mathbf{e}_i)$ is contained in the relative interior of σ'.

We say that Procedure 4.2.11 terminates *positively* if after completing Step 1 and Step 2 it is possible to determine the coordinates of all the other vertices of $\mathcal{S}$ in Step 3. If Procedure 4.2.11 terminates positively then $\Gamma \subset \operatorname{trop} X$ and the containment induces the subdivision $\mathcal{S}$ on T. If this is not the case then this does not imply that Γ is not contained in $\operatorname{trop} X$. In fact Γ might be contained in $\operatorname{trop} X$ but with induced subdivision different from $\mathcal{S}$.

Let T be a tree with at most $n+1$ labelled leaves and $\mathcal{S}$ be a subdivision of T. We define $C_{\mathcal{S}}^T$ by

$$C_{\mathcal{S}}^T = \{p \in \operatorname{trop} \mathbb{G}(1,n) : T_{\Gamma_p} = T,\ \mathcal{S} \text{ is the subdivision of } T \text{ induced by } \Gamma_p \subset \operatorname{trop} X\},$$

this is the set of points in $\operatorname{trop}^{\text{ext}} \mathbb{G}(1,n)$ that correspond to tropical lines contained in $\operatorname{trop} X$ and such that their containment induces the subdivision $\mathcal{S}$ of the graph T_Γ.

Remark 4.2.12. The subdivision $\mathcal{S}$ in $C^T_{\mathcal{S}}$ has to satisfy Properties 4.2.8 and 4.2.9 since it is the subdivision induced by the containment $\Gamma \in C^T_{\mathcal{S}}$.

Each $C^T_{\mathcal{S}}$ is contained in $\mathrm{F}_1(\operatorname{trop} X)$ hence

$$\bigcup_{T,\mathcal{S}} C^T_{\mathcal{S}} \subset \mathrm{F}_1(\operatorname{trop} X) \tag{4.4}$$

where the union is over all trees T with at most $n+1$ labelled leaves and $\mathcal{S}$ is a subdivision of T.

The following Lemma 4.2.13 proves equality in (4.4).

Lemma 4.2.13. *Let Γ be a tropical line. Then Γ is contained in* $\operatorname{trop} X$ *if and only if there exists a labelled subdivision $\mathcal{S}$ with Properties 4.2.8 and 4.2.9 for which the procedure above terminates positively. Moreover $\mathcal{S}$ is the labelled subdivision of T_Γ induced by $\Gamma \subset \operatorname{trop} X$.*

Proof. We prove here the second part of the statement since the first follows from the discussion above.

Assume $\mathcal{S}'$ is the labelled subdivision induced by $\Gamma \subset \operatorname{trop} X$. We have to show that this is equal to $\mathcal{S}$. Firstly for each vertex $\overline{V} \in T_\Gamma$ then $V \in \Gamma$ is in the relative interior of a unique cell of $\operatorname{trop} X$ hence the label of $\overline{V}$ has to be the same in $\mathcal{S}$ and $\mathcal{S}'$.

Let e be an edge of T_Γ with end points $(\overline{V}, \sigma)$ and $(\overline{V}', \sigma')$. Suppose e is subdivided in $\mathcal{S}'$ then this is also subdivided in $\mathcal{S}$. In fact let $\tau_e = VV'$ then

$$VV' = (\tau_e \cap \sigma_1) \cup \ldots \cup (\tau_e \cap \sigma_s)$$

and $(\tau_e \cap \sigma_i) \subset \sigma_i^\circ$ has dimension 1 for any i. The edge e is replaced in $\mathcal{S}$ by the path $(\overline{V}, \sigma)(\overline{V_1}, \sigma_1') \ldots (\overline{V_s}, \sigma_s')(\overline{V}', \sigma')$. Each vertex V_i is the point of intersections of $\sigma_i \cap \sigma_{i+1}$ with the line parallel to the span of $\mathbf{v}$ and passing through V, where $\mathbf{v}$ is the direction of e. Each cell $\sigma_i' \subset \operatorname{trop} X$ is a face of $\sigma_i \cap \sigma_{i+1}$ and it contains V_i in its relative interior. Hence there is no cell of $\operatorname{trop} X$ that contains τ_e in its relative interior hence e has to be subdivided in $\mathcal{S}$ too. Since Procedure 4.2.11 terminates positively we also have that the subdivision of e in $\mathcal{S}$ has to be the same as the one in $\mathcal{S}'$. On the other hand if e is not subdivided in $\mathcal{S}'$ then it is also not subdivided in $\mathcal{S}$. Step 3 can not be completed since the relative interior of τ_e is contained in the smallest cell that has σ and σ' as faces. □

Propositions 4.2.14 and 4.2.15 show that the closure of $C^T_{\mathcal{S}}$ is a polyhedron and its closure is the union of $C_{\mathcal{S}'}^{T'}$ for some tree T' and subdivision $\mathcal{S}'$.

Proposition 4.2.14. *The set $C_{\mathcal{S}}^{T}$ is either a linear space or it is the relative interior of a polyhedron.*

Proof. Let p be any point in $\operatorname{trop}^{\text{ext}} \mathbb{G}(1,n) \subset \operatorname{trop} \mathbb{P}^{\binom{n+1}{2}-1}$ then by Lemma 4.2.13 we have that $p \in C_{\mathcal{S}}^{T}$ if and only if $T_{\Gamma_p} = T$ and Procedure 4.2.11 terminates positively using $\mathcal{S}$ as input.

We prove that this happens if and only if some particular polyhedral conditions (i.e. linear equalities and strict inequalities) are satisfied by the coordinates p_{ij} of p. This implies that $C_{\mathcal{S}}^{T}$ is either a linear space, if we only have linear equations, or it is the relative interior of a polyhedron (Remark 2.1.10).

Firstly we observe that $T_{\Gamma_p} = T$ if and only if p is contained in the relative interior of the cone C_T of $\operatorname{trop} \mathbb{G}(1,n)$. Moreover Procedure 4.2.11 terminates positively if and only if each point $V \in \Gamma_p$ is contained in the relative interior of the cell that labels $\overline{V}$ in $\mathcal{S}$. Hence the coordinates of V have to satisfy any linear equalities and strict inequalities that define the relative interior of this cell. These are all polyhedral conditions on the $p_{i,j}$.

We show that for every cone C_T and for every vertex $\overline{V} \in T$ there exist linear forms $\{f_V^0, f_V^1 \dots\}$ such that coordinates of $\overline{V}$ in Γ_p are $(f_V^0(p_{ij}), f_V^1(p_{ij}), \dots)$ where the p_{ij}'s are the coordinates of p. This implies that if all cells labelling $\mathcal{S}$ are linear spaces and C_T is a linear space then $C_{\mathcal{S}}^{T}$ is a linear space too. In all other cases $C_{\mathcal{S}}^{T}$ is the relative interior of a polyhedron (see Remark 2.1.10).

The coordinates of all points V associated to vertices of $\mathcal{S}$ can be computed from p using the two graphs T_Γ and $\mathcal{S}$. We distinguish two cases: vertices V such that $\overline{V} \in V(T_\Gamma)$ and vertices V such that $\overline{V} \in V(\mathcal{S}) \setminus V(T_\Gamma)$.

In the first case let M be the rank 2 matroid on $\{0, \dots, m\}$ such that $C_T \subset \operatorname{trop} \operatorname{Gr}_M$. Suppose M is loop free and let P_M be the matroid polytope associated to M (Definition 2.4.6). The tropical line Γ_p is a subcomplex Σ of the dual complex to the regular subdivision Δ_p of P_M and we also have a correspondence between vertices of Σ and vertices of T_Γ. The vertices of Σ are dual to maximal facets of Δ_p hence applying the definition of regular subdivision it is possible to compute the coordinates of these vertices as in Remark 2.1.18. In fact if F is the face dual to V then there exists $v_{n+1} > 0$ such that V is solution to $(\mathbf{x}, v_{n+1}) \cdot (\mathbf{e}_{ij}, 1) = p_{ij}$ for $\mathbf{e}_{ij} \in F$. Since every $p \in C_T$ induce the same subdivision we have that this system is solvable even when we consider p_{ij} as parameters. The solution will be $\mathbf{x} = (f_V^0(p_{ij}), \dots, f_V^m(p_{ij}))$ where f_V^i are linear forms. Now consider W such that $\overline{W} \in V(\mathcal{S}) \setminus V(T_\Gamma)$. Let e be an edge of T_Γ that has been replaced in $\mathcal{S}$ with a path whose vertices are $\overline{V}, \overline{V}_1, \dots, \overline{V}_s, \overline{V}'$, with $\overline{V}$ and $\overline{V'}$ end points of e and $\overline{W} = \overline{V}_i$ for some $i = 1, \dots, s$. The coordinates of the vertices $V_1, \dots, V_s$ are computed as in Step 3 of Procedure 4.2.11. Hence these are also linear expression in the Plücker coordinates and they have to satisfy the strict inequalities and equalities defining the relative interior of the cone that labels the corresponding vertex.

If M is not loop free we consider the cone C'_T of $\mathrm{Gr}_{U_{2,n'+1}}$ that is the image of C_T via the linear isomorphism in (4.2). There is a bijection l_ψ between the tropical lines in C_T and in C'_T. Hence we compute the coordinates of the vertices of the tropical lines in C'_T as in the case of a loop free matroid and then applying the map l_ψ we get the vertices of the tropical lines in C_T.

If the support of Γ is a classical line then it is the line dual to a codimension 1 face F of P_M. The equation of the line is $\mathbf{x} \cdot \mathbf{e}_{ij} = p_{ij}$ for $\mathbf{e}_{ij} \in F$. The coordinates of the points corresponding to vertices of $\mathcal{S}$ are then obtained intersecting the line with the cones that label the vertices. □

Proposition 4.2.15. *Assume $C^T_{\mathcal{S}}$ is not a classical linear space and let $\mathcal{F}$ be a face of $\overline{C^T_{\mathcal{S}}}$. Then there exist a tree T' and a subdivision $\mathcal{S}'$ of T' such that $\mathcal{F} = \overline{C^{T'}_{\mathcal{S}'}}$.*

In Proposition 4.2.14 we saw that the equalities and strict inequalities defining $C^T_{\mathcal{S}}$ come from conditions imposing that the vertices of any $\Gamma \in C_T$ have to be contained in the relative interior of some cells of $\operatorname{trop} X$. Suppose $C^T_{\mathcal{S}}$ is the relative interior of a polyhedron then a face of $\overline{C^T_{\mathcal{S}}}$ is obtained by intersecting $\overline{C^T_{\mathcal{S}}}$ with the hyperplanes associated to some strict inequalities defining $C^T_{\mathcal{S}}$. The key point in the proof of Lemma 4.2.15 is to interpret this new set of equalities and strict inequalities as conditions on the vertices of tropical lines contained in $\operatorname{trop} X$. Before proving Lemma 4.2.15 we compute an example of $\overline{C^T_{\mathcal{S}}}$.

Example 4.2.16. Consider the tropical plane $\operatorname{trop} L$ of Example 4.2.10 and let T and $\mathcal{S}$ be as in Figure 4.3. We denote by τ_i the ray $\operatorname{pos}(\mathbf{e}_i)$ in $\operatorname{trop} L$, by $\overline{V_{i,j}}$ the vertex labelled by $\tau_{i,j}$ and by $\overline{V_0}$ the vertex labelled by $\mathbf{0}$. Note that $V_0 = (0,0,0)$. Using the proof of Proposition 4.2.14 and Procedure 4.2.11 we can explicitly describe $C^T_{\mathcal{S}} \subset \operatorname{trop}(\mathbb{G}(1,3) \cap T^5) \subset \mathbb{R}^6/\mathbb{R}\mathbf{1}$. Firstly we need to have $C^T_{\mathcal{S}} \subset C_T$, that is

$$p_{02} + p_{13} = p_{03} + p_{12} < p_{01} + p_{23}.$$

The coordinates of the non-leaf vertices are the following (cf. [MS15, Example 4.3.19]):

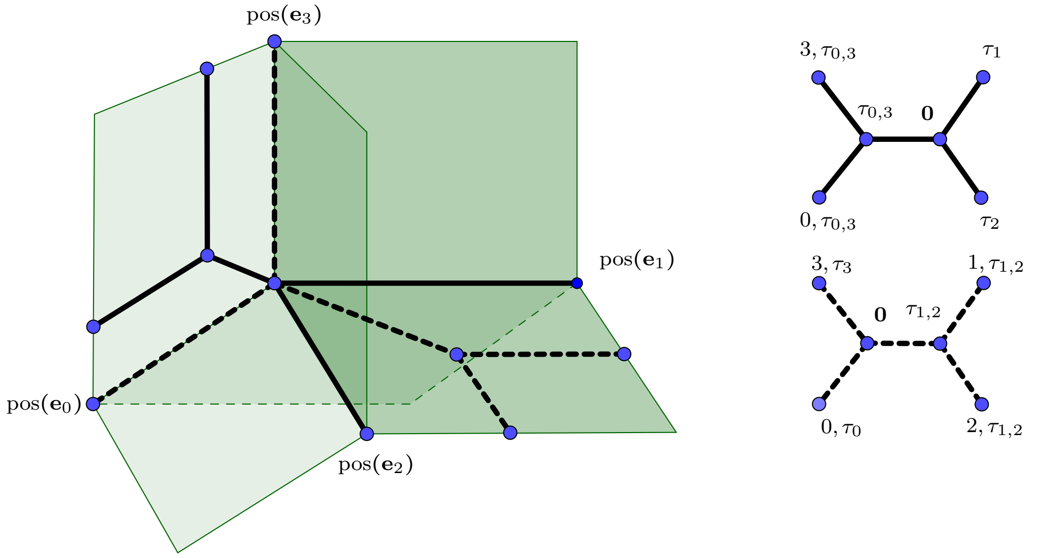

Figure 4.4: Lines contained in two faces of $\overline{C_{\mathcal{S}}^T}$ and their associated subdivisions as in Example 4.2.16.

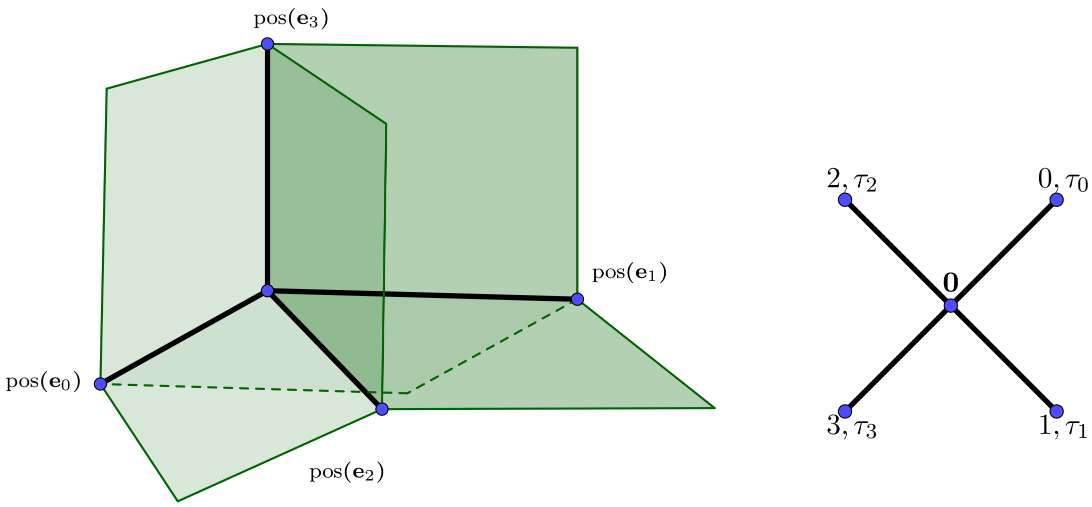

Figure 4.5: The tropical line Γ associated to the point $(0, 0, 0, 0, 0, 0)$ and its subdivision $\mathcal{S}$ as in Example 4.2.16.

$$V_{0,3} = (p_{02} - p_{23}, p_{12} - p_{23}, p_{12} - p_{13})$$
$$V_{1,2} = (p_{01} - p_{13}, p_{01} - p_{03}, p_{12} - p_{13}).$$

Since $V_{i,j}$ has to be in the relative interior of $\tau_{i,j}$ for $\{i,j\} \in \{\{0,3\},\{1,2\}\}$ we get that

$$C^T_{\mathcal{S}} = \{p \in \mathbb{R}^6/\mathbb{R}\mathbf{1} : p_{23} - p_{13} > 0, p_{01} - p_{13} > 0,\ p_{02} = p_{12} = p_{13} = p_{03}\}.$$

This is the relative interior of the 2-dimensional cone C spanned by rays pos$(-1,-1,-1,-1,-1,0)$ and pos$(1,0,0,0,0,0)$ in $\mathbb{R}^6/\mathbb{R}\mathbf{1}$. The set $\overline{C^T_{\mathcal{S}}} \smallsetminus C^T_{\mathcal{S}}$ is given by three faces: the two rays of C and the point $(0,0,0,0,0,0)$. A point in ray pos$(-1,-1,-1,-1,-1,0)$ (*resp.* in ray pos$(1,0,0,0,0,0)$) is associated to the solid (*resp.* dotted) tropical line in Figure 4.4. The point $(0,0,0,0,0,0)$ corresponds to the tropical line in Figure 4.5.

The following lemma is the first step towards proving Lemma 4.2.15.

Let C_T and $C_{T'}$ be two cones of trop Gr_M such that $C_{T'}$ is a facet of C_T and M is a rank two matroid on $\{0,1,\dots,m\}$. The tree T' is obtained from T by contracting an edge e to a new vertex V. Denote by $\overline{V_1}$ and $\overline{V_2}$ the end points of e. For any vertex $\overline{Q}$ in T let $f_0^{\overline{Q}},\dots,f_m^{\overline{Q}}$ be the linear forms associated to $\overline{Q}$ in C_T as defined in the proof of Proposition 4.2.14.

Lemma 4.2.17. *Let $\overline{W} \neq \overline{V_1}, \overline{V_2}$ be vertices of T. Then for every $p' \in C_{T'}$ the vertex $W' \in \Gamma_{p'}$ associated to $\overline{W}$ has coordinates $(f_0^{\overline{W}}(p'),\dots,f_m^{\overline{W}}(p'))$. The vertex V in $\Gamma_{p'}$ corresponding to $\overline{V}$ has coordinates $(f_0^{\overline{V_1}}(p'),\dots,f_m^{\overline{V_1}}(p')) = (f_0^{\overline{V_2}}(p'),\dots,f_m^{\overline{V_2}}(p'))$.*

Proof. Suppose M is loop free. We denote by Δ_T (*resp.* $\Delta_{T'}$) the regular subdivision induced by points of C_T (*resp.* $C_{T'}$) on the matroid polytope P_M.
The subdivision Δ_T is obtained from $\Delta_{T'}$ by subdividing the facet $F \in \Delta_{T'}$, corresponding to the contracted edge in T', into two facets F_1 and F_2, with $F_1 \cap F_2$ a facet of both $F_i's$. The vertices dual to F, F_1 and F_2 are $\overline{V}, \overline{V}_1$ and $\overline{V}_2$ respectively. Let $\overline{W} \neq \overline{V_1}, \overline{V_2}$. Then for any $p \in C_T$ we have $W = (f_1^{\overline{W}}(p),\dots,f_m^{\overline{W}}(p))$ and by definition of regular subdivision we have that

$$(f_0^{\overline{W}}(p),\dots,f_m^{\overline{W}}(p)) \cdot \mathbf{e}_{i,j} = p_{ij} \text{ for } \mathbf{e}_{i,j} \in F' \qquad (4.5)$$
$$(f_0^{\overline{W}}(p),\dots,f_m^{\overline{W}}(p)) \cdot \mathbf{e}_{i,j} < p_{ij} \text{ for } \mathbf{e}_{i,j} \in P_T \smallsetminus F' \qquad (4.6)$$

where $\mathbf{e}_{i,j} = \mathbf{e}_i + \mathbf{e}_j$ are the vertices of P_M and F' is the facet of Σ_{P_T}(and of $\Sigma_{P_{T'}}$) dual to W. Since $C_{T'}$ is in the closure of C_T then for p' in $C_{T'}$ we have

$$(f_0^{\overline{W}}(p'), \dots, f_m^{\overline{W}}(p')) \cdot \mathbf{e}_{i,j} = p'_{ij} \text{ for } \mathbf{e}_{i,j} \in F' \tag{4.7}$$

$$(f_0^{\overline{W}}(p'), \dots, f_m^{\overline{W}}(p')) \cdot \mathbf{e}_{i,j} \leq p'_{ij} \text{ for } \mathbf{e}_{i,j} \in P_T \smallsetminus F'. \tag{4.8}$$

We claim that all the inequalities in (4.8) are strict. In fact if this was not the case we would have a face of $\Sigma_{P_{T'}}$ that strictly contains F and this is not possible since F is a maximal face. Hence $(f_0^{\overline{W}}(p'), \dots, f_m^{\overline{W}}(p'))$ is the point W'. Consider now $\overline{V} \in T'$. Applying the same method as above we have that

$$(f_0^{\overline{V_1}}(p')), \dots, f_m^{\overline{V_1}}(p')) \cdot \mathbf{e}_{i,j} = p_{ij} \text{ for } \mathbf{e}_{i,j} \in (F_1 \cap F_2) \cup (F_1 \smallsetminus (F_1 \cap F_2)) \tag{4.9}$$

$$(f_0^{\overline{V_1}}(p')), \dots, f_m^{\overline{V_1}}(p')) \cdot \mathbf{e}_{i,j} \leq p_{ij} \text{ for } \mathbf{e}_{i,j} \in P_T \smallsetminus F_1 \tag{4.10}$$

and

$$(f_0^{\overline{V_2}}(p'), \dots, f_m^{\overline{V_2}}(p')) \cdot \mathbf{e}_{i,j} = p_{ij} \text{ for } \mathbf{e}_{i,j} \in (F_1 \cap F_2) \cup (F_2 \smallsetminus (F_1 \cap F_2)) \tag{4.11}$$

$$(f_0^{\overline{V_2}}(p'), \dots, f_m^{\overline{V_2}}(p')) \cdot \mathbf{e}_{i,j} \leq p_{ij} \text{ for } \mathbf{e}_{i,j} \in P_T \smallsetminus F_2. \tag{4.12}$$

The points $(f_0^{\overline{V_1}}(p'), \dots, f_m^{\overline{V_1}}(p'))$ and $(f_0^{\overline{V_2}}(p'), \dots, f_m^{\overline{V_2}}(p'))$ are both dual to the facets of the subdivision induced by p'. If all inequalities in (4.10) and in (4.12) are all equalities then T had only one edge that has been contracted to $V = (f_0^{\overline{V_2}}(p'), \dots, f_m^{\overline{V_2}}(p')) = (f_0^{\overline{V_1}}(p'), \dots, f_m^{\overline{V_1}}(p'))$. Suppose this is not the case then the inequalities in (4.10) (*resp.* (4.12)) are equalities for $\mathbf{e}_{i,j} \in F_2 \smallsetminus F_1$ (*resp.* $\mathbf{e}_{i,j} \in F_1 \smallsetminus F_2$). In fact the facets of P_T dual to $(f_0^{\overline{V_2}}(p'), \dots, f_m^{\overline{V_2}}(p'))$ and $(f_0^{\overline{V_1}}(p'), \dots, f_m^{\overline{V_1}}(p'))$ both strictly contain F and this implies that these points are dual to F. Hence we have $V = (f_0^{\overline{V_1}}(p'), \dots, f_m^{\overline{V_1}}(p')) = (f_0^{\overline{V_2}}(p'), \dots, f_m^{\overline{V_2}}(p'))$.

If M is not loop free we consider the cones C'_T and $C'_{T'}$ of $\mathrm{Gr}_{U_{2,n'+1}}$ that are the images of C_T and $C_{T'}$ via the linear isomorphism in (4.2). There is a bijection l_ψ between the tropical lines in C_T and in C'_T. Hence we apply the lemma of the tropical lines in C'_T and $C_{T'}$ and then applying the map l_ψ we get the result for the tropical lines in $C_{T'}$.

□

Proof of Proposition 4.2.15. It suffices to prove this for facets of $\overline{C_{\mathcal{S}}^T}$.
We assume that $C_{\mathcal{S}}^T$ is the relative interior of a polyhedron (if it is a linear space then $\overline{C_{\mathcal{S}}^T} = C_{\mathcal{S}}^T$).We show that the relative interior of a facet F of $\overline{C_{\mathcal{S}}^T}$ is equal to $C_{\mathcal{S}'}^{T'}$

for some T' and $\mathcal{S}'$ so that $F = \overline{C^{T'}_{\mathcal{S}'}}$.

The relative interior of F is obtained by intersecting $C^T_{\mathcal{S}}$ with the hyperplane associated to one strict inequality defining $C^T_{\mathcal{S}}$. From Procedure 4.2.11 and Proposition 4.2.14 we have that this inequality is either defining the relative interior of the cone C_T where $C^T_{\mathcal{S}}$ is contained or it is imposing that a vertex $(\overline{V}, \sigma) \in \mathcal{S}$ is in the relative interior of the cell σ. It is also possible to have both situations for a single inequality. For this reason we distinguish two cases based on which tree is associated to $\mathcal{F}^\circ$:

I: $\mathcal{F}^\circ \subset C_T$;

II: $\mathcal{F}^\circ \subset C_{T'} \prec C_T$ with $C_{T'}$ a facet of C_T.

Case I

Since $\mathcal{F}^\circ \subset C_T$ we have that the inequality that is now an equality is one that is imposing that some vertices $\overline{W_1}, \dots, \overline{W_l} \in \mathcal{S}$ have to be in the relative interior of cells $\sigma_1, \dots, \sigma_l$ respectively. This implies that for every $i = 1, \dots, l$ in each line $\Gamma \in \mathcal{F}^\circ$ the point $W_i \in \Gamma$ is contained in a face τ_i of σ_i. Hence Γ is still contained in $\operatorname{trop} X$ but the containment induces a new subdivision $\mathcal{S}'$ of T. In order to obtain $\mathcal{S}'$ we first change the label of the $\overline{W_i}$'s. We denote by $\mathcal{S}_1$ the labelled graph obtained in this way.

Property 4.2.8 is satisfied by $\mathcal{S}_1$. Let $(\overline{V}, \sigma)$ and $(\overline{V'}, \sigma')$ be the end points of an edge of $\mathcal{S}$ and assume the new label of $\overline{V}$ is τ. Since τ is a face of σ we have that τ and σ' will still be two faces of a cell of $\operatorname{trop} X$.

Property 4.2.9 might not be satisfied by $\mathcal{S}_1$. Let e be an edge of $\mathcal{S}_1$ subdivided in a path with vertices $\overline{V}, \overline{V_1}, \dots, \overline{V_s}, \overline{V}'$. There can be two vertices $\overline{W} = \overline{V}_i$ and $\overline{W'} = \overline{V}_{i+1}$ in the path that have the same label. This means that either $W = W'$ in every line Γ in $\mathcal{F}^\circ$ or the closed segment WW' is contained in the relative interior of a cell of $\operatorname{trop} X$. In the first case we replace e in $\mathcal{S}_1$ with the path $\overline{V}\ \overline{V_1} \dots \overline{V}_{i-1}\ \overline{V_i}\ \overline{V}_{i+2} \dots \overline{V_s}\ \overline{V}'$ and with $\overline{V_1} \dots \overline{V}_{i-1}\ \overline{V}_{i+2} \dots \overline{V_s}$ in the other. We denote the new subdivision $\mathcal{S}_2$. If $\mathcal{S}_2$ does not satisfies Property 4.2.9 then we apply the above procedure recursively to $\mathcal{S}_2$. In this way we obtain a subdivision $\mathcal{S}'$ that satisfies Properties 4.2.8 and 4.2.9. Hence we have that $\mathcal{F}^\circ \subset C^T_{\mathcal{S}'}$. The other inclusion follows from the fact that the equalities and inequalities defining $C^T_{\mathcal{S}'}$ are the same as the ones defining $\mathcal{F}^\circ$.

Case II

We have $\mathcal{F}^\circ \subset C^{T'}$ where $C^{T'}$ is a facet of C^T. The tree T' is obtained from T by contracting an edge e. Let $\mathcal{S}'$ be the subdivision of T' obtained from $\mathcal{S}$ by contracting the edge e or the path replacing e with a vertex $\overline{V}$. Consider on every vertex $\overline{V'} \in \mathcal{S}'$ different from $\overline{V}$ the label of $\overline{V'}$ in $\mathcal{S}$ and let $\overline{V}$ be labelled by the maximal dimensional common face of σ_1 and σ_2 where $(\overline{V_1}, \sigma_1)$ and $(\overline{V_2}, \sigma_2)$ are the end points of e. We prove that all tropical lines in $\mathcal{F}^\circ$ are contained in $\operatorname{trop} X$ and the induced subdivision is the same for all of them. Let Γ be a tropical line in $\mathcal{F}$. Then we apply Procedure 4.2.11 with the subdivision $\mathcal{S}'$. First we observe that by Lemma 4.2.17 we have that for any $\overline{W} \in T'$ the linear forms defining the coordinates of W in any $\Gamma \in C_{T'}$ are the same as the ones defining W in any tropical line in C_T. This implies that also the linear forms defining the coordinates of the other vertices of $\mathcal{S}'$ are the same as the ones of $\mathcal{S}$. Hence we can read the strict inequalities and equalities of $\mathcal{F}$ as imposing to the vertices of Γ corresponding to vertices of $\mathcal{S}'$ to be contained in some cells of $\operatorname{trop} X$. Then we have that any line in $\mathcal{F}^\circ$ is contained in $\operatorname{trop} X$. This inclusion might not induce the subdivision $\mathcal{S}'$. If Γ is a tropical line in $\mathcal{F}$ and $W \in \Gamma$ corresponds to $(\overline{W}, \sigma) \in \mathcal{S}'$ then W has to be contained either in σ° or in the relative interior of a face τ of σ since one of the strict inequalities defining σ might be changed to an equality. Consider $\mathcal{S}''$ where $(\overline{W}, \sigma)$ has been replace with $(\overline{W}, \tau)$ for any such $\overline{W}$. This subdivision satisfies Property 4.2.8. If $\mathcal{S}''$ satisfies Property 4.2.9 then $\mathcal{F}^\circ \subset C^{T'}_{\mathcal{S}''}$ otherwise we proceed as in the case $\mathcal{F} \subset C^T$ to substitute $\mathcal{S}''$ with the correct subdivision $\mathcal{S}'''$. As in the previous case the containment $C^{T'}_{\mathcal{S}''} \subset \mathcal{F}^\circ$ (*resp.* $C^{T'}_{\mathcal{S}'''} \subset \mathcal{F}^\circ$) follows from the fact that the equalities and strict inequalities defining $C^{T'}_{\mathcal{S}''}$ (*resp.* $C^{T'}_{\mathcal{S}'''}$) are the same as the ones defining $\mathcal{F}^\circ$. □

Example 4.2.18. In Example 4.2.16 we have that $\operatorname{pos}(-1,-1,-1,-1,-1,0)$ and $\operatorname{pos}(1,0,0,0,0,0)$ are faces of $\overline{C^T_{\mathcal{S}}}$ that are contained in C_T. This is Case I of the proof of Proposition 4.2.15. An example of Case II is the point $(0,0,0,0,0,0)$ that is contained in $C_{T'}$ with T' as in Figure 4.5.

We are now ready to prove Theorem 4.2.2.

Proof of Theorem 4.2.2. We have proved that

$$\mathrm{F}_1(\operatorname{trop} X) = \bigcup_{\mathcal{S},T} C^T_{\mathcal{S}}$$

where the union is over all trees T associated to the cones of $\operatorname{trop} \mathbb{G}(1,n) \cap \mathcal{O}$ and over all subdivision $\mathcal{S}$ of T that satisfy Properties 4.2.8 and 4.2.9.

By Lemma 4.2.15 we deduce that

$$\mathrm{F}_1(\operatorname{trop} X) = \bigcup_{\mathcal{S},T} \overline{C^T_{\mathcal{S}}}.$$

We need to show that the intersection of any two of these polyhedra is either empty or a face of both.

We have that $\overline{C^T_{\mathcal{S}}}^\circ = C^T_{\mathcal{S}}$ since $C^T_{\mathcal{S}}$ is either a linear space or it is the relative interior of a polyhedron. If $(T,\mathcal{S}) \neq (T',\mathcal{S}')$ then the relative interiors of $\overline{C^T_{\mathcal{S}}}$ and $\overline{C^{T'}_{\mathcal{S}'}}$ do not intersect. In fact let $T \neq T'$ then $C^\circ_T \cap C^\circ_{T'} = \varnothing$ and hence $C^T_{\mathcal{S}} \cap C^{T'}_{\mathcal{S}}$ is empty. On the other hand if $T = T'$ and $\mathcal{S} \neq \mathcal{S}'$ then $C^T_{\mathcal{S}}$ and $C^T_{\mathcal{S}'}$ do not intersect by Lemma 4.2.13.

Hence we deduce that if $\overline{C^T_{\mathcal{S}}} \cap \overline{C^{T'}_{\mathcal{S}'}}$ is not empty then there exists two faces $F \subset \overline{C^T_{\mathcal{S}}}$ and $F' \subset \overline{C^{T'}_{\mathcal{S}'}}$ such that $F^\circ \cap (F')^\circ \neq \varnothing$. By Lemma 4.2.15 we have that $F = C^{T_1}_{\mathcal{S}_1}$ and $F' = C^{T_2}_{\mathcal{S}_2}$. This implies that $C^{T_1}_{\mathcal{S}_1} = C^{T_2}_{\mathcal{S}_2}$ hence $F = F'$. Using again Lemma 4.2.13 we deduce that $\overline{C^T_{\mathcal{S}}}$ and $\overline{C^{T'}_{\mathcal{S}'}}$ intersect in a face. □

Remark 4.2.19. It is not necessary to have a surjective valuation val. Let $G = \mathrm{val}(\mathbb{K})$ be the value group of val and assume $G \subsetneq \mathbb{R}$. Then for any variety $X \subset \mathbb{P}^n$ we have that each face of $\operatorname{trop} X$ is a $\mathrm{val}(\mathbb{K})$-polyhedron hence it is defined by linear equalities and inequalities with coefficients in $\mathrm{val}(\mathbb{K})$. In particular if Γ is a tropical line then the coordinates of its vertices are in $\mathrm{val}(\mathbb{K})$. This implies that the set $\overline{C^T_{\mathcal{S}}} \in \mathbb{R}^{m+1}/\mathbb{R}\mathbf{1} \cong \mathbb{R}^m$ is not a polyhedron but it is the intersection of a polyhedron with $\mathrm{val}(\mathbb{K}^*)^m$. Let $\mathcal{O}$ be an orbit of $\operatorname{trop} \mathbb{P}^{\binom{n+1}{2}-1}$ then we can define $\mathrm{F}_1(\operatorname{trop} X) \cap \mathcal{O}$ to be the euclidean closure of $\bigcup_{T,\mathcal{S}} \overline{C^T_{\mathcal{S}}}$ with $C_T \in \mathcal{O}$.

The structure of the tropical Fano scheme is strictly connected to the structure of the tropical variety $\operatorname{trop} X$.

Proposition 4.2.20. *Let $\mathcal{O} = \operatorname{trop} \mathbb{G}(1,n) \cap O$ where O is a torus orbit of $\mathbb{P}^{\binom{n+1}{2}-1}$ and let O' be the unique orbit of $\mathbb{P}^n$ that contains the lines parametrized by O. If $\operatorname{trop}(X \cap O')$ is a fan then $\mathrm{F}_1(\operatorname{trop} X) \cap \mathcal{O}$ is fan.*

Proof of Proposition 4.2.20. Let Γ be the tropical line associated to the point p. We denote by Γ_λ the tropical line associate to $\lambda p = (\lambda \cdot p_{ij})$ and we have that $T_\Gamma = T_{\Gamma_\lambda}$ since $\operatorname{trop} \mathbb{G}(1,n)$ is a fan.

We have to show that if p is a point in $\mathrm{F}_1(\operatorname{trop} X) \cap \mathcal{O}$ then λp is in $\mathrm{F}_1(\operatorname{trop} X) \cap \mathcal{O}$ for $\lambda \geq 0$.

If $\lambda = 0$ then by [MS15, Theorem 4.4.5] the support of Γ_0 is the recession fan of Γ. Since $\Gamma \subset \operatorname{trop} X$ then all the unbounded edges of Γ are contained in $\operatorname{trop}(X \cap O')$ then also the corresponding rays in Γ_0 are in $\operatorname{trop} X$. This implies Γ_0 is contained in $\operatorname{trop} X$.

Assume now $\lambda > 0$. We have that $T_{\Gamma_\lambda} = T_\Gamma$. We claim that $\Gamma_\lambda \subset \operatorname{trop} X$ and the induced subdivision in $\mathcal{S}$. We use Procedure 4.2.11 and we check that we can complete each of the steps. Step 1 can be completed since $T_\Gamma = T_{\Gamma_\lambda}$ hence they have the same leaves.

Given any vertex $(\overline{V}, \sigma)$ in T_Γ we have seen in Lemma 4.2.15 that $\overline{V} \in \Gamma$ is the vector dual to a maximal face of the subdivision of the matroid polytope P. Since the Plücker vector of Γ_λ is $\lambda \cdot p_{ij}$ then $V_\lambda = \lambda V$. This implies that the conditions in Step 2 are satisfied since V_λ is also contained in σ°.

Let $(\overline{V}, \sigma)(\overline{V_1}, \sigma_1), \ldots, (\overline{V_s}, \sigma_s)(\overline{V'}, \sigma')$ be ordered vertices of a path in $\mathcal{S}$ that replaces an edge $\overline{VV'}$ of T. We can compute the coordinates of the associated points $(V_1)_\lambda, \ldots, (V_s)_\lambda$ in Γ_λ as in Step 3 of Procedure 4.2.11. We have again that $(V_i)_\lambda = \lambda \cdot V_i$. Since $\lambda \cdot V_i$ is also contained in σ_i then also Step 3 is completed. □

Remark 4.2.21. The result in Proposition 4.2.20 holds even in the case $\operatorname{trop} X = p + \Sigma$ where Σ is the fan $\Sigma = \operatorname{trop}(X \star P)$ with $\operatorname{val}(P) = p$ (see [MS15, Proposition 5.5.11]). In fact from the definition of the tropical Fano scheme we have $\mathrm{F}_1(\operatorname{trop} X) = \mathrm{F}_1(\Sigma + p) = \mathrm{F}_1(\Sigma) + p$.

Note that $\operatorname{trop} \mathrm{F}_1(X)$ does not have the same property described in Proposition 4.2.20. There are varieties X such that $\operatorname{trop} X$ is a fan but $\operatorname{trop} \mathrm{F}_1(X)$ is not. In the next section we compute an explicit example of this (Example 4.3.6).

4.3 Linear spaces and generic hypersurfaces

In this section we show that there exist linear spaces and hypersurfaces for which the containment $\operatorname{trop} \mathrm{F}_1(X) \subset \mathrm{F}_1(\operatorname{trop} X)$ is strict. In Theorem 4.3.3 we prove that for a generic plane L in $\mathbb{P}^5$ there exists a tropical line in $\operatorname{trop} L$ that is not realizable in L. We then compute an explicit example of a plane $L \subset \mathbb{P}^5$ with this property and we show that $\dim \operatorname{trop} \mathrm{F}_1(L) < \dim \mathrm{F}_1(\operatorname{trop} L)$. Finally in Proposition 4.3.7 we prove that the containment is strict for a generic hypersurface X that tropicalizes to the standard tropical hyperplane.

Definition 4.3.1 ([Nic18]). A semialgebraic subset of an algebraic variety X is a subset of X that can locally be defined by finitely many Boolean operators and inequalities of the form $\operatorname{val}(f) \leq \operatorname{val}(g)$ where f, g are algebraic functions on X.

Remark 4.3.2. Note that every set U that is Zariski open is also a semialgebraic set.

Theorem 4.3.3. *There exists a semi-algebraic set in* $\mathbb{G}(2,5)$ *whose points are planes* $L \subset \mathbb{P}^5$ *such that* $\operatorname{trop} \mathrm{F}_1(L) \subsetneq \mathrm{F}_1(\operatorname{trop} L)$.

Proof. Let $\mathcal{L}$ be the standard tropical plane in $\operatorname{trop} \mathbb{P}^5$. This is the closure in $\operatorname{trop} \mathbb{P}^5$ of the tropicalization of the uniform matroid $U_{3,6}$ ([MS15, Example 4.2.13]), that is the fan in $\mathbb{R}^5 \cong \mathbb{R}^6/\mathbb{R}\mathbf{1}$ given by the six 2-dimensional cones $\text{pos}(\mathbf{e}_i, \mathbf{e}_j)$ for $0 \leq i < j \leq 5$. Let $\Gamma^\circ \subset \text{trop}(T^5) \cong \mathbb{R}^6/\mathbb{R}\mathbf{1}$ be the 1-dimensional fan given by rays $\text{pos}(\mathbf{e}_0 + \mathbf{e}_1), \text{pos}(\mathbf{e}_2 + \mathbf{e}_3), \text{pos}(\mathbf{e}_4 + \mathbf{e}_5)$. The closure of Γ° in $\text{trop}\,\mathbb{P}^5$ is a tropical line Γ and since $\Gamma^\circ \subset \mathcal{L} \cap \text{trop}\,T^5$ then Γ is contained in $\mathcal{L}$.

Given $p \in \mathbb{G}(2,5)$ we denote by L_p the associated plane in $\mathbb{P}^5$. We now show that we can find an open semi-algebraic set $\mathcal{U}$ in $\mathbb{G}(2,5)$ such that for every $p \in \mathcal{U}$ we have $\text{trop}\,L_p = \mathcal{L}$ and there does not exist $l \subset L_p$ such that $\text{trop}\,l = \Gamma$.

Firstly we have that $\text{trop}\,L_p = \mathcal{L}$ if and only if $p \in \mathcal{U} = \{q \in \mathbb{G}(2,5) : \text{val}(q) = (0,\dots,0)\}$. Let $x_0,\dots,x_5$ be the coordinates of $\mathbb{P}^5$ and denote by L_p the plane associated to a point $p \in \mathbb{G}(2,5)$. The plane L_p induces a line arrangement $\mathcal{A} = \{l_0,\dots,l_5\} \subset \mathbb{P}^5$ given by the lines $l_i = L_p \cap \{x_i = 0\}$.

Let $w_{i,j}$ be the point of intersection of l_i and l_j. There exists a Zariski open set $\mathcal{V}$ of $\mathbb{G}(2,5)$ such that for every $p \in \mathcal{V}$ the line arrangement satisfies the following conditions

(I) $l_i \cap l_j \cap l_s = \varnothing$ for any three distinct indices i,j,s;

(II) $w_{i_0,i_1}, w_{i_2,i_3}, w_{i_4,i_5}$ are not collinear unless $\{i_0,i_1\} \cap \{i_2,i_3\} \subset \{i_4,i_5\}$.

Note that condition (I) is satisfied by all linear spaces L_p with $p \in \mathcal{U}$ since $(0,\dots,0) \in \text{trop}(\mathbb{G}(2,5) \cap T^{19})$ (see Theorem 2.4.1). Let $\mathcal{U} = \mathcal{V} \cap \{p \in \mathbb{G}(2,5) : \text{val}(p) = (0,\dots,0)\}$. This is a semi-algebraic set and we now prove that if $p \in \mathcal{U}$ then Γ is not realisable in L_p. Consider the three points that are the closure of the three rays of Γ° in $\text{trop}\,\mathbb{P}^5$. By Theorem 2.5.9 we have that these are contained in the orbits $\mathcal{O}_{0,1}, \mathcal{O}_{2,3}, \mathcal{O}_{4,5}$ of $\text{trop}\,\mathbb{P}^5$ where $\mathcal{O}_{i,j} = \text{trop}\,O_{i,j} = \text{trop}(\{[x_0,\dots,x_5] \in \mathbb{P}^5 : x_i = x_j = 0\})$. Moreover these are the points $\text{trop}\,w_{0,1}, \text{trop}\,w_{2,3}, \text{trop}\,w_{4,5}$ since $\text{trop}(L_p \cap O_{i,j}) = \text{trop}\,L_p \cap \mathcal{O}_{i,j}$. Suppose there exists a line $l \subset L_p$ such that $\text{trop}\,l = \Gamma$. Then the line l contains $w_{0,1}, w_{2,3}$ and $w_{4,5}$ since $\text{trop}\,w_{0,1}, \text{trop}\,w_{2,3}, \text{trop}\,w_{4,5}$ are contained in Γ. This is a contradiction since L_p satisfies condition (II). □

In the following we produce an explicit example of a plane $L \in \mathbb{P}^5$ with $\text{trop}\,\text{F}_1(L) \subsetneq \text{F}_1(\text{trop}\,L)$ and we compute the tropical Fano scheme $\text{F}_1(\text{trop}\,L)$. In fact for tropical linear spaces the tropical Fano scheme is the tropical prevariety defined by the tropicalization of the classical incidence relations ([Haq12, Theorem 1]).

Example 4.3.4. Let $\mathbb{K}$ be the field of generalised Puiseux series $\mathbb{C}((\mathbb{R}))$ and $K = \mathbb{C}$.

Let L be the plane defined by

$$98x_1 + 80x_3 - 128x_4 - 461x_5 = 0$$
$$98x_2 - 74x_3 - 58x_4 - 153x_5 = 0$$
$$98x_0 - 52x_3 - 54x_4 + 13x_5 = 0.$$

The line arrangement $\mathcal{A} = \{l_i = L \cap \{x_i = 0\} : i = 0, \dots, 5\}$ satisfies conditions (I) and (II) in the proof of Theorem 4.3.3 hence the matroid associated to L is the uniform matroid $U_{3,6}$. The coordinates of the point $p \in \mathbb{G}(2,5)$ associated to L are non zero complex numbers hence $\operatorname{val}(p) = (0, \dots, 0) \in \operatorname{trop}(G(2,5) \cap T^{19})$. This implies that $\operatorname{trop} L = \mathcal{L}$. The Fano scheme is defined by the ideal

$$\begin{aligned}
&(49p_{25} - 37p_{35} - 29p_{45}, 49p_{15} + 40p_{35} - 64p_{45}, 49p_{05} - 26p_{35} - 27p_{45},\\
&98p_{24} - 74p_{34} + 153p_{45}, 98p_{14} + 80p_{34} + 461p_{45},\\
&98p_{04} - 52p_{34} - 13p_{45}, 98p_{23} + 58p_{34} + 153p_{35},\\
&98p_{13} + 128p_{34} + 461p_{35}, 98p_{03} + 54p_{34} - 13p_{35},\\
&98p_{12} + 144p_{34} + 473p_{35} + 73p_{45}, 98p_{02} + 10p_{34} - 91p_{35} - 92p_{45},\\
&98p_{01} - 112p_{34} - 234p_{35} - 271p_{45})
\end{aligned}$$

The tropicalization $\operatorname{trop} F_1(L)$ is a 2–dimensional fan in $\operatorname{trop} \mathbb{P}^9$.

The tropical Fano scheme $\mathrm{F}_1(\operatorname{trop} L)$ is the tropical prevariety defined by the tropical incidence relations associated to $\operatorname{trop} L$. These are given by the Plücker relations generating $\mathbb{G}(1,5)$ and by all tropical polynomials of the form

$$\bigoplus_{i \in T \smallsetminus S} p_{S \cup i} p_{T \smallsetminus i}$$

where $S \subset \{0,1,2,3\} = T$, $|S| = 1$ and $p_{T \smallsetminus i}$ are the valuations of coordinates of p. In this case $p_{T \smallsetminus i} = 0$ for all $0 \leq i \leq 3$. The explicit classical polynomials whose tropicalization are the tropical incidence relations can be found in Appendix.

Computations show that while $\operatorname{trop} F_1(L)$ is a 2-dimensional fan the tropical Fano scheme $\mathrm{F}_1(\operatorname{trop} L)$ is a fan with 15 maximal cones of dimension 3 and 30 maximal cones of dimension 2. The rays of $\mathrm{F}_1(\operatorname{trop} L)$ are the same as the rays of $\operatorname{trop} F_1(L)$ and the dimension 2 maximal cones are also cones of $\operatorname{trop} F_1(L)$. The dimension 3 cones of $\mathrm{F}_1(\operatorname{trop} L)$ are the ones parametrizing tropical lines Γ whose associated tree is a snowflake (see Figure 4.6). In fact if the centre of Γ is in the origin of $\operatorname{trop} \mathbb{P}^5$ then $\Gamma \subset \operatorname{trop} L$ since $\operatorname{trop} L$ is given by the cones $\operatorname{pos}(\mathbf{e}_i, \mathbf{e}_j)$ for any $i \neq j$. In Figure 4.6 we have an example of one of these tropical lines and its position in $\operatorname{trop} L$.

In the next example we show that it is possible to realise the line Γ in the proof of Theorem 4.3.3 by choosing a particular L' with $\operatorname{trop} L' = \mathcal{L}$.

Example 4.3.5. Let $L' \subset \mathbb{P}^5$ be the plane defined by $23x_0 - 6x_1 - x_2 - x_4 = 0, -8x_0 + 2x_1 - x_3 + x_4 = 0, -97x_0 + 25x_1 + x_3 + 3x_5 = 0$. As in the previous example the line arrangement $\mathcal{A}'$ associated to L' satisfies condition (I) of the proof of Theorem 4.3.3 and we have $\operatorname{trop} L' = \mathcal{L} = \operatorname{trop} L$. However $\mathcal{A}'$ does not satisfy condition (II). Let $p'_{i,j}$ be the point $L \cap O_{i,j}$ with $O_{i,j}$ the orbit of $\mathbb{P}^5$ where $x_i = x_j = 0$. The points $p'_{0,1}, p'_{2,3}$ and $p'_{4,5}$ are collinear and the line ℓ passing through them is defined by the following equations

$$x_4 - x_5 = 0, 3x_2 - x_3 = 0, 3x_1 + 4x_3 + 12x_5 = 0, 3x_0 + x_3 + 3x_5 = 0.$$

The tropical line $\operatorname{trop} \ell$ is the fan with rays $\operatorname{pos}(\mathbf{e}_0 + \mathbf{e}_1), \operatorname{pos}(\mathbf{e}_2 + \mathbf{e}_3), \operatorname{pos}(\mathbf{e}_4 + \mathbf{e}_5)$. Hence this is the tropical line Γ of the proof of Theorem 4.3.3. The ideal associated to the Fano scheme $F_1(L')$ is

$$\begin{aligned}
&(6p_{25} - 2p_{35} - p_{45}, 6p_{15} + 8p_{35} + 97p_{45}, 6p_{05} + 2p_{35} + 25p_{45}\\
&6p_{24} - 2p_{34} - p_{45}, 6p_{14} + 8p_{34} + 73p_{45}, 6p_{04} + 2p_{34} + 19p_{45}\\
&6p_{23} + p_{34} - p_{35}, 6p_{1,3} - 97p_{34} + 73p_{35},\\
&6p_{03} - 25p_{34} + 19p_{35}, 6p_{12} - 31p_{34} + 23p_{35} - 4p_{45},\\
&6p_{02} - 8p_{34} + 6p_{35} - p_{45}, 6p_{01} + p_{34} - p_{35} - 3p_{45})
\end{aligned}$$

and $\operatorname{trop} \mathrm{F}_1(L')$ is a 2–dimensional fan in $\mathbb{R}^{10}/\mathbb{R}\mathbf{1}$. Both $\operatorname{trop} \mathrm{F}_1(L')$ and $\operatorname{trop} \mathrm{F}_1(L)$ are contained in $\mathrm{F}_1(\operatorname{trop} L)$ since $\operatorname{trop} L = \operatorname{trop} L'$. All rays of $\operatorname{trop} \mathrm{F}_1(L)$ are also rays of $\operatorname{trop} \mathrm{F}_1(L')$ but this has also an extra ray r. The tropical lines associated to points in r have the snowflake in Figure 4.6 as associated tree. Moreover let $C = \operatorname{pos}(r_1, r_2, r_3)$ be one of the 3-dimensional cones of $\mathrm{F}_1(\operatorname{trop} L)$. We have that $C \cap \operatorname{trop} F_1(L)$ is given by the two dimensional faces of C. On the other hand $C \cap \operatorname{trop} F_1(L')$ is the union of the three cones $\operatorname{pos}(r_1, r_1 + r_2 + r_3), \operatorname{pos}(r_2, r_1 + r_2 + r_3), \operatorname{pos}(r_3, r_1 + r_2 + r_3)$ (see Figure 4.7).

The explicit computations for the tropical varieties and prevarieties are done with Tropical.m2 [AKL+17], while we use `Polymake` [GJ] and the *Polyhedra* package in `Macaulay2` [GS] to get the tree associated the tropical lines in a cone of $\mathrm{F}_1(\operatorname{trop} L)$.

In Proposition 4.2.20 we proved that if $\operatorname{trop} X$ is a fan then $\mathrm{F}_1(\operatorname{trop} X)$ is a fan. This is not the case for $\operatorname{trop} \mathrm{F}_1(X)$. In Example 4.3.6 we exhibit a plane L''

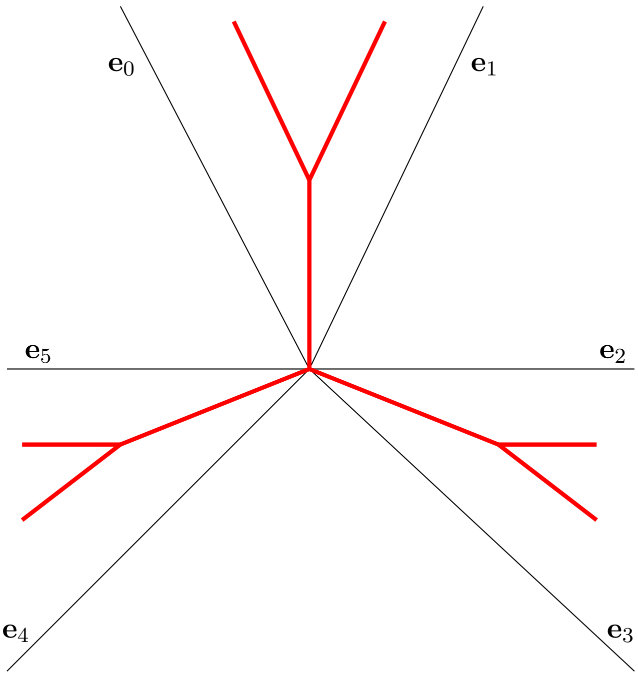

Figure 4.6: A tropical line contained in the three cones $\mathrm{pos}(\mathbf{e}_0, \mathbf{e}_3), \mathrm{pos}(\mathbf{e}_2, \mathbf{e}_5), \mathrm{pos}(\mathbf{e}_1, \mathbf{e}_4)$ of $\operatorname{trop} L \subset \mathbb{R}^5 \cong \mathbb{R}^6/\mathbb{R}\mathbf{1}$ as in Example 4.3.4.

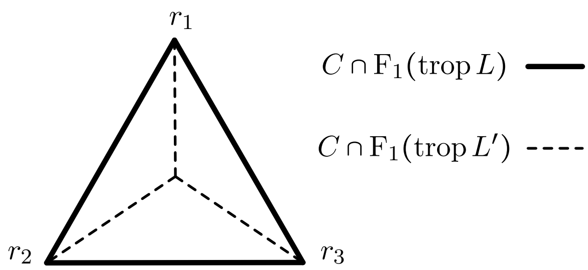

Figure 4.7: A section of the cone $C \subset \mathrm{F}_1(\operatorname{trop} L)$.

such that $\operatorname{trop} L''$ is a fan but $\operatorname{trop} \mathrm{F}_1(L'')$ is not.

Example 4.3.6. Consider $\mathbb{K}$ to be the field of generalised Puiseux series $\mathbb{C}((\mathbb{R}))$. Let L'' be the plane in $\mathbb{P}^5$ spanned by the rows of the following matrix

$$M = \begin{pmatrix} 1 & 1 & 0 & t & 1 & 1 \\ 1 & t+1 & 1 & 2 & t & 0 \\ 5 & 8 & 6 & 9 & 7 & 10 \end{pmatrix}.$$

The line spanned by the first two rows of M tropicalizes to a tropical line whose associated tree is a snowflake as in Figure 4.6. The corresponding point in $\operatorname{trop} F_1(L'')$ is $\mathbf{e}_{01} + \mathbf{e}_{23} + \mathbf{e}_{45}$ in $\mathcal{O} = \operatorname{trop}(\mathbb{G}(1,5) \cap T^{\binom{6}{2}}) \subset \mathbb{R}^{\binom{6}{2}}/\mathbb{R}\mathbf{1}$, where the $\mathbf{e}_{ij}$'s denote the standard basis vectors of $\mathbb{R}^{\binom{6}{2}}$. Suppose the ray $\operatorname{pos}(\mathbf{e}_{01} + \mathbf{e}_{23} + \mathbf{e}_{45})$ is contained in $\operatorname{trop}(\mathrm{F}_1(L'') \cap T^{\binom{6}{2}})$. Then the closure of the ray $\operatorname{pos}(\mathbf{e}_{01} + \mathbf{e}_{23} + \mathbf{e}_{45})$ in $\operatorname{trop} \mathbb{P}^{\binom{6}{2}}$ is a point Q and it is contained in $\operatorname{trop} \mathrm{F}_1(L'')$. In fact $\operatorname{trop} \mathrm{F}_1(L'')$ is the closure of $\operatorname{trop}(\mathrm{F}_1(L'') \cap T^{\binom{6}{2}})$ in $\operatorname{trop} \mathbb{P}^{\binom{6}{2}}$ (see [MS15, Theorem 6.2.18]). The point Q is the origin in $\operatorname{trop} O = \mathbb{R}^{12}/\mathbb{R}\mathbf{1}$ with $O = \{[p_{ij}] \in \mathbb{P}^{\binom{6}{2}-1} : p_{01} = p_{23} = p_{45} = 0\}$. Hence Γ_Q is the tropical line given by the fan in $\mathbb{R}^6/\mathbb{R}\mathbf{1}$ with rays $\operatorname{pos}(\mathbf{e}_0 + \mathbf{e}_1), \operatorname{pos}(\mathbf{e}_2 + \mathbf{e}_3)$ and $\operatorname{pos}(\mathbf{e}_4 + \mathbf{e}_5)$. We now show that this tropical line is not realizable in L'' hence $\operatorname{trop}(\mathrm{F}_1(L)) \cap T^{\binom{6}{2}-1}$ is not a fan even though $\operatorname{trop}(L \cap T^5)$ is a fan.

Firstly the matroid associated to L'' is $U_{3,6}$. In fact the line arrangement $\mathcal{A} = \{L \cap O_{i,j} : 0 \leq i < j \leq 5\}$, where $O_{i,j} = \{[x_0 : \ldots : x_5] \in \mathbb{P}^5 : x_i = x_j = 0$ and $x_k \neq 0$ for $k \neq i, k \neq j\}$, satisfies condition (I) of proof of Theorem 4.3.3. Moreover $\mathcal{A}$ also satisfies condition (II) hence the points p_{ij} of intersection of these lines are not collinear. The tropical line Γ_Q passes through $\operatorname{trop}(p''_{01}), \operatorname{trop}(p''_{23})$ and $\operatorname{trop}(p''_{45})$ hence it is not realizable in L'' otherwise p''_{01}, p''_{23} and p''_{45} would be collinear.

Another instance where the containment $\operatorname{trop} \mathrm{F}_1(X) \subset \mathrm{F}_1(\operatorname{trop} X)$ is strict is the case of *general* hypersurfaces whose tropicalization has the same support of a tropical linear space. In [BVdV79, Theorem 8] the authors show that there is an open set in the space of hypersurfaces in $\mathbb{P}^n$ for which the Fano scheme of lines has dimension $2n - d - 3$. We call such hypersurfaces *general*.

Proposition 4.3.7. *If X is a general hypersurface of degree $d > 1$ and the tropicalization $\operatorname{trop} X$ has the same support as a tropicalized linear space then we have that $\operatorname{trop} \mathrm{F}_1(X) \subsetneq \mathrm{F}_1(\operatorname{trop} X)$.*

Proof. If L is a $(n-1)$-dimensional linear space then the dimension of $\mathrm{F}_1(L)$ is $\dim \mathbb{G}(2,n) = 2n - 4$. By hypothesis we have that $\mathrm{F}_1(\operatorname{trop} X) = \mathrm{F}_1(\operatorname{trop} L)$ and $\dim \mathrm{F}_1(\operatorname{trop} L) \geq \dim \operatorname{trop} F_1(L) = 2n - 4$. On the other hand the dimension of

$\operatorname{trop} F_1(X)$ is equal to the dimension of $\mathrm{F}_1(X)$ which is $2n - d - 3$. Suppose $\operatorname{trop} F_1(X) = \mathrm{F}_1(\operatorname{trop} X)$ then we would have $2n - d - 3 \geq 2n - 4$ but this is not the case if $d > 1$. □

4.4 Toric varieties

In this section we look at the Fano scheme of lines in toric varieties. We prove that in this case the tropical Fano scheme is equal to the tropicalization of the classical Fano scheme.

Consider a toric variety X associated to a set of lattice points $\mathcal{A} = \{\mathbf{a}_0, \ldots, \mathbf{a}_n\} \subset \mathbb{Z}^m \times \{1\}$ and let A be the matrix whose columns are the points in $\mathcal{A}$. The variety X has a natural embedding in $\mathbb{P}^n$ given by a monomial map $\phi_{\mathcal{A}} : T^m \times \mathbb{K}^* \to \mathbb{P}^n$ (see [CLS11, Section 2.1]). We denote the image of this map by $X_{\mathcal{A}}$. The matrix A also defines a map $\operatorname{trop} \phi_{\mathcal{A}} : \mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$ and by [MS15, Theorem 3.2.13] we have that the tropicalization of $X_{\mathcal{A}} \cap T^n \subset \mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ is the quotient by $\mathbb{R}\mathbf{1}$ of the image of $\operatorname{trop} \phi_{\mathcal{A}}$ which is the classical linear space spanned by the rows of A. Since the embedding of the toric variety only depends on the row span of A ([CLS11, Proposition 1.1.9]) it is possible to recover the ideal defining $X_{\mathcal{A}}$ from $\operatorname{trop}(X_{\mathcal{A}} \cap T^n)$.

Example 4.4.1. Let $X_{\mathcal{A}} \subset \mathbb{P}^3$ be the toric variety associated to the set of lattice points $\mathcal{A} = \{(1,1,1), (0,0,1), (0,-1,1), (1,0,1)\}$. The matrix A is

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

and the ideal defining $X_{\mathcal{A}}$ is $(xz - yw)$. The tropicalization $\operatorname{trop}(X_{\mathcal{A}} \cap T^3)$ is the set $(\{(x,y,z,w) : x + z = y + w\})/\mathbb{R}\mathbf{1}$ hence it is the quotient by $\mathbb{R}\mathbf{1}$ of the linear span of the rows of A.

By contrast with the case of linear spaces we show that for toric varieties the tropical Fano scheme is the same as the tropicalization of the classical Fano scheme.

Theorem 4.4.2. *Let $X = X_{\mathcal{A}}$ be a toric variety. Then* $\mathrm{F}_1(\operatorname{trop} X) = \operatorname{trop} \mathrm{F}_1(X)$.

We prove this result by showing that for each tropical line $\Gamma \subset \operatorname{trop} X$ there exists a line $\ell \subset X$ that tropicalizes to it. We construct the line ℓ using *Cayley structures* on $\mathcal{A}$. We use results in [IZ17, Section 3] where Ilten and Zotine prove that for each Cayley structure τ there exists a subvariety Z_τ of $\mathrm{F}_1(X_{\mathcal{A}})$ and from τ

it is possible to deduce equations of the lines parametrised by Z_τ.

Given a set of $n+1$ lattice points $\mathcal{A}$ in $\mathbb{Z}^s \times \{1\}$, let L be the kernel of the map defined by the matrix A. If $\mathbf{l} \in L$ we can write $\mathbf{l} = \sum_{l_i>0} l_i \mathbf{e}_i - \sum_{l_i<0} -l_i \mathbf{e}_i$ and denote by $l^+ = \sum_{l_i>0} l_i \mathbf{e}_i$ and $l^- = \sum_{-l_i<0} -l_i \mathbf{e}_i$. We have that $\mathbf{l} \in L$ if and only if $\sum l_i \mathbf{a}_i = 0$. The toric variety $X_\mathcal{A} \subset \mathbb{P}^n$ is generated by binomials of the form $\mathbf{x}^{l^+} - \mathbf{x}^{l^-} = \prod_{l_i>0} x_i^{l_i} - \prod_{l_i<0} x_i^{l_i}$ with $\mathbf{l} \in L$ ([MS15, Proposition 1.1.9]).

A face τ of $\mathcal{A}$ is the intersection of a face of $\operatorname{conv}(\mathcal{A})$ with $\mathcal{A}$. Denote by Δ_s the standard basis of $\mathbb{Z}^{s+1}$. Using these notations we give the definition of Cayley structure:

Definition 4.4.3. An s-Cayley structure on τ is a surjective map $\pi : \tau \to \Delta_s$ such that if $\mathbf{l} \in L$ and $\{i : l_i \neq 0\} = \{i : \mathbf{a}_i \in \tau\}$ then $\sum_{l_i \neq 0} l_i \pi(\mathbf{a}_i) = 0$ or equivalently $\sum_{l_i>0} l_i \pi(\mathbf{a}_i) = \sum_{l_i<0} -l_i \pi(\mathbf{a}_i)$ with $\mathbf{a}_i \in \tau$.

Example 4.4.4. Consider the set of lattice points $\mathcal{A}$ as in Example 4.4.1. A 1–Cayley structure is given by $\pi : \mathcal{A} \to \mathbb{Z}^2$ with $\pi((0,0,1)) = \pi((0,-1,1)) = \mathbf{e}_1$ and $\pi((1,0,1)) = \pi((1,1,1)) = \mathbf{e}_2$. An example of a surjective map $\pi : \mathcal{A} \to \Delta_1$ that is not a Cayley structure is given by $\pi : \mathcal{A} \to \mathbb{Z}^2$ with $\pi((1,1,1)) = \pi((0,-1,1)) = \mathbf{e}_1$ and $\pi((0,0,1)) = \pi((1,0,1)) = \mathbf{e}_2$. We can see that $\mathbf{l} = (1,-1,1,-1)$ is in L hence $(1,1,1) - (0,0,1) + (0,-1,1) - (1,0,1) = 0$ but if we apply π we get $2\mathbf{e}_1 - 2\mathbf{e}_2 = 0$ which is a contradiction.

We now prove that given a tropical line in $\operatorname{trop}(X \cap T^n)$ then it is possible to associate to it a Cayley structure on $\mathcal{A}$.

Proposition 4.4.5. *Let $X_\mathcal{A} \subset \mathbb{P}^n$ be a toric variety and let Γ be a tropical line contained in $\operatorname{trop}(X_\mathcal{A} \cap T^n)$. If Γ has $s+1$ unbounded edges then it is possible to associate to Γ an s–Cayley structure on $\mathcal{A}$.*

The following is a technical lemma which will be used for the proof of Proposition 4.4.5.

Lemma 4.4.6. *Let $\Gamma \subset \mathbb{R}^{m+1}/\mathbb{R}\mathbf{1}$ be a tropical line and let $\mathcal{V}$ be the set of vectors in $\mathbb{R}^{m+1}/\mathbb{R}\mathbf{1}$ parallel to the s unbounded edges of Γ. If the support of Γ is a classical line spanned by $\mathbf{w} \in \mathbb{R}^{m+1}/\mathbb{R}\mathbf{1}$ consider $\mathcal{V} = \{\mathbf{w}\}$. Then there is a choice of representatives $\{\mathbf{v}^1, \dots, \mathbf{v}^s\}$ for each vectors in $\mathcal{V}$ such that*

(i) v_i^j is either 1 or 0 for every $j = 1, \dots, s$ and $i = 0, \dots, m$;

(ii) for any $i = 0, \dots, n$ either $v_i^j = 0$ for all $j = 1, \dots, s$ or there exists a unique $\mathbf{v}_j$ such that $v_i^j = 1$.

Proof. Consider the recession fan F of Γ. Its rays are given by $\mathrm{pos}(\mathbf{v}_1), \dots, \mathrm{pos}(\mathbf{v}_s)$, hence it is sufficient to prove the Lemma in the case where $\Gamma = F$. Let ℓ be the classical line such that $\mathrm{trop}\,\ell = \Gamma$ then using Theorem 4.2.6 in [MS15] we can describe Γ as the fan constructed from the lattice of flats of the matroid M_ℓ associated to ℓ. If $F_1, \dots, F_s$ are the non-empty minimal elements in the lattice then $\mathrm{pos}(\mathbf{e}_{F_1}), \dots, \mathrm{pos}(\mathbf{e}_{F_s})$ are the rays of Γ, where $\mathbf{e}_{F_t} = \sum_{j \in F_t} \mathbf{e}_j$. From this we deduce that the possible vectors in $\{\mathbf{v}_1, \dots, \mathbf{v}_s\}$ are all vectors whose coordinates are 0 or 1. This concludes point (i).

To prove point (ii) we observe that for any $i = 1, \dots, n+1$ we have that either i is not contained in any F_t or there exists a unique t such that $(\mathbf{e}_{F_t})_i = 1$ and $(\mathbf{e}_{F_t})'_i = 0$ for $i' \neq i$. In fact if there are F_t and $F_{t'}$ such that $(\mathbf{e}_{F_t})_i = (\mathbf{e}_{F_{t'}})_i = 1$ then $F_t \cap F_{t'} \neq \varnothing$. This contradicts the minimality of F_t and $F_{t'}$ since $F_t \cap F_{t'}$ would be a non-empty flat contained in both F_t and $F_{t'}$. □

Proof of Proposition 4.4.5. Let A be the matrix associated to $\mathcal{A} = \{\mathbf{a}_0, \dots, \mathbf{a}_n\}$. Since $\mathcal{A} \subset \mathbb{Z}^m \times \{1\}$ we have that the last row of A is given by $(1, 1, \dots, 1)$.

Let Γ be a tropical line in $\mathrm{trop}(X \cap T^n)$. All the unbounded edges of Γ are contained in $\mathrm{trop}\,X \cap T^n \subset \mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ hence the vectors $\mathbf{v}_0, \dots, \mathbf{v}_s$ parallel to them are part of a set of generators for the linear space $\mathrm{trop}\,X \cap T^n$. By balancing condition and Lemma 4.4.6 we have that $\mathbf{v}_0 + \dots + \mathbf{v}_s = (1, \dots, 1) \in \mathbb{R}^n/\mathbb{R}\mathbf{1}$. Since $(1, \dots, 1)$ is the last row of A we can assume that $\mathbf{v}_1, \dots, \mathbf{v}_s$ are the first s rows of A. Since there is a unique $\mathbf{v}_i$ with last coordinate equal to 1 and we can assume that it is $\mathbf{v}_0$. The columns of A are the points of $\mathcal{A}$ and by Lemma 4.4.6 they can be partitioned in $s+1$ sets $A_0, \dots, A_s$. The set A_i, for $i = 0, \dots, s-1$, is given by the points where the i-th coordinate is 1 and the rest of the first s entries are zero. The set A_s is given by the points whose first s coordinates are all zero. We define the map $\pi : \mathcal{A} \to \Delta_s$ by sending the points in A_r to $\mathbf{e}_r \in \mathbb{Z}^{s+1}$. This map is an s–Cayley structure on $\mathcal{A}$. In fact let $\mathbf{l} \in L$ with $\mathbf{l} = l^+ - l^- = \sum_{l_i > 0} l_i \mathbf{e}_i - \sum_{l_i < 0} l_i \mathbf{e}_i$ and $\{i : l_i \neq 0\} = \{i : \mathbf{a}_i \in \mathcal{A}\}$ then we have

$$\sum_{l_i > 0, \mathbf{a}_i \in A_0} l_i \mathbf{a}_i + \dots + \sum_{l_i > 0, \mathbf{a}_i \in A_s} l_i \mathbf{a}_i = \sum_{l_i < 0, \mathbf{a}_i \in A_0} -l_i \mathbf{a}_i + \dots + \sum_{l_i < 0, \mathbf{a}_i \in A_s} -l_i \mathbf{a}_i.$$

We need to prove that

$$\sum_{l_i > 0, \mathbf{a}_i \in A_0} l_i \pi(\mathbf{a}_i) + \dots + \sum_{l_i > 0, \mathbf{a}_i \in A_s} l_i \pi(\mathbf{a}_i) = \sum_{l_i < 0, \mathbf{a}_i \in A_0} -l_i \pi(\mathbf{a}_i) + \dots + \sum_{l_i < 0, \mathbf{a}_i \in A_s} -l_i \pi(\mathbf{a}_i). \tag{4.13}$$

We have that

$$\sum_{l_i>0,\mathbf{a}_i\in A_0} l_i\pi(\mathbf{a}_i) + \ldots + \sum_{l_i>0,\mathbf{a}_i\in A_s} l_i\pi(\mathbf{a}_i) = (\sum_{l_i>0,\mathbf{a}_i\in A_0} l_i, \ldots, \sum_{l_i>0,\mathbf{a}_i\in A_s} l_i)$$

and

$$\sum_{l_i<0,\mathbf{a}_i\in A_0} -l_i\pi(\mathbf{a}_i) + \ldots + \sum_{l_i<0,\mathbf{a}_i\in A_s} -l_i\pi(\mathbf{a}_s) = (\sum_{l_i<0,\mathbf{a}_i\in A_0} -l_i, \ldots, \sum_{l_i<0,\mathbf{a}_i\in A_s} -l_i).$$

Let P be the point $\sum_{l_i>0,\mathbf{a}_i\in A_0} l_i\mathbf{a}_i + \ldots + \sum_{l_i>0,\mathbf{a}_i\in A_s} l_i\mathbf{a}_i = \sum_{l_i<0,\mathbf{a}_i\in A_0} -l_i\mathbf{a}_i + \ldots + \sum_{l_i<0,\mathbf{a}_i\in A_s} -l_i\mathbf{a}_i$. The first coordinate of P is given by the first coordinate of $\sum_{l_i>0,\mathbf{a}_i\in A_0} l_i\mathbf{a}_i$, that is $\sum_{l_i>0,\mathbf{a}_i\in A_0} l_i$, or equivalently by the first coordinate of $\sum_{l_i<0,\mathbf{a}_i\in A_0} -l_i\mathbf{a}_i$, that is $\sum_{l_i<0,\mathbf{a}_i\in A_0} -l_i$.

From this we obtain $\sum_{l_i>0,\mathbf{a}_i\in A_0} l_i = \sum_{l_i<0,\mathbf{a}_i\in A_0} -l_i$. In the same way we have $\sum_{l_i>0,\mathbf{a}_i\in A_1} l_i = \sum_{l_i<0,\mathbf{a}_i\in A_1} -l_i$, $\ldots$, $\sum_{l_i>0,\mathbf{a}_i\in A_{s-1}} l_i = \sum_{l_i<0,\mathbf{a}_i\in A_{s-1}} -l_i$. Using these equalities we also have $\sum_{l_i>0,\mathbf{a}_i\in A_s} l_i = \sum_{l_i<0,\mathbf{a}_i\in A_s} -l_i$ therefore

$$(\sum_{l_i>0,\mathbf{a}_i\in A_0} l_i, \ldots, \sum_{l_i>0,\mathbf{a}_i\in A_s} l_i) = (\sum_{l_i>0,\mathbf{a}_i\in A_0} l_i, \ldots, \sum_{l_i>0,\mathbf{a}_i\in A_s} l_i)$$

hence the equality in 4.13 is satisfied. □

We illustrate a Cayley structure associated to a tropical line in the following example.

Example 4.4.7. Let $\mathcal{A}$ be the set given by the columns of the matrix A where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 7 & 3 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

The toric variety $X_{\mathcal{A}}$ is defined by the ideal $(x_2x_3 - x_4^2) \subset \mathbb{C}[x_0, x_1, x_2, x_3, x_4]$. The tropical line Γ_1 spanned by $(0,1,0,0,0)$ is contained in $\mathrm{trop}(X_{\mathcal{A}} \cap T^3)$. We can define a 1–Cayley structure associated to Γ_1 sending $(0,1,2,1)$,$(0,0,7,1)$, $(0,0,3,1)$ and $(0,0,5,1)$ to $\mathbf{e}_1$ and $(1,0,1,1)$ to $\mathbf{e}_2$.

We also notice that the tropical line Γ_2 whose rays are $\mathrm{pos}(1,0,0,0,0)$, $\mathrm{pos}(0,1,0,0,0)$ and $\mathrm{pos}(-1,-1,0,0,0)$ is contained in $\mathrm{trop}(X_{\mathcal{A}} \cap T^3)$. The 2–Cayley structure associated to Γ_1 is the map sending $(0,1,2,1)$ to $\mathbf{e}_1$, $(1,0,1,1)$ to $\mathbf{e}_2$, $(0,0,7,1)$, $(0,0,3,1)$ and $(0,0,5,1)$ to $\mathbf{e}_3$.

Proof of Theorem 4.4.2. We will prove that given a tropical line $\Gamma \subset \operatorname{trop} X$ there exists a line L' in X such that $\operatorname{trop} L' = \Gamma$. If Γ is one of the boundary lines of $\operatorname{trop} \mathbb{P}^n$ then this will be the tropicalization of a line in $X \cap O$ where O is the corresponding one dimensional orbit of $\mathbb{P}^n$.

Assume now that Γ is in $\operatorname{trop}(X \cap O)$ with O an orbit of $\mathbb{P}^n$ with dimension greater than 1. We can consider $Y = X \cap \overline{O}$ as a subvariety of $\overline{O} \cong \mathbb{P}^s$ with $s = \dim \overline{O}$. The variety Y is also a toric variety and we denote by $\mathcal{A}'$ the set of lattice points associated to it.

Suppose Γ has $l+1$ unbounded edges. By Lemma 4.4.5 we have that there exists a l–Cayley structure π on $\mathcal{A}'$. Let Z_π be the subvariety of $\mathrm{F}_k(Y)$ associated to π (see [IZ17, Section 3, Section 4]). This is the closed torus orbit of the linear space L generated by $\mathbf{v}_0, \ldots, \mathbf{v}_l \in \mathbb{R}^{\dim Y+1}$ where

$$\mathbf{v}_i^j = \begin{cases} 1 \text{ if } \pi(\mathbf{a}_j) = \mathbf{e}_i \\ \quad 0 \text{ else.} \end{cases}$$

Let Γ' be the translation of Γ to the origin then there exists p a point in $\operatorname{trop} Y$ such that $\Gamma = \Gamma' + p$. The set of vectors parallel to the rays of Γ generate a linear space $\mathcal{L}$ and $\Gamma \subset \mathcal{L} + p$. We have that $\mathcal{L} = L$. In fact by definition of the $\mathbf{v}_i^j$ and by construction of π in Lemma 4.4.5 the matrix $\begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_l \end{pmatrix}$ is equal to the sub-matrix of A given by the first l rows. The equations of L are $\operatorname{codim} L$ binomials of type $x_i - x_j$ hence $\operatorname{trop} L = L$. There exists $t \in T^{\dim Y}$ such that $\operatorname{trop}(t \cdot L) = \mathcal{L} + p$. We will now show that Γ is the tropicalization of a line in $t \cdot L$ hence in Y. First observe that we can choose $x_0, \ldots, x_l \in \{x_0, \ldots, x_s\}$ such that for any $q \in L$ we have $q_i \in \{x_0, \ldots, x_l\}$. This implies that the projection $\phi = \phi_{x_0,\ldots,x_l} : \mathbb{P}^s \to \mathbb{P}^l$ induces an isomorphism between L and $\mathbb{P}^l$. Let ψ^{-1} be its inverse. Since ϕ and ϕ^{-1} are linear monomial maps then $\operatorname{trop}(\phi) = \phi$ and $\operatorname{trop}(\phi^{-1}) = \phi^{-1}$. Consider the line $\phi(\Gamma') \subset \operatorname{trop} \mathbb{P}^l$. This line is realizable in $\mathbb{P}^l$, that is there exists $\ell' \in \mathbb{P}^l$ such that $\operatorname{trop} \ell' = \phi(\Gamma')$. Now $\phi^{-1}(\ell') \subset L \subset Y$ and $\operatorname{trop}(\phi^{-1}(\ell')) = \operatorname{trop}(\phi^{-1})(\operatorname{trop}(\ell')) = \Gamma'$. If we consider $\ell = t \cdot \phi(\ell')$ then $\operatorname{trop}(\ell) = \Gamma$.

□

Example 4.4.8. Consider the toric variety $X_\mathcal{A}$ of Example 4.4.7. We use the proof of Theorem 4.4.2 to compute the lines ℓ_1, ℓ_2 in X_Γ that tropicalize to Γ_1 and Γ_2 respectively. The line ℓ_1 is the line L associated to the 1–Cayley structure π_1. Its defining equations are $x_0 - x_2 = 0, x_2 - x_3 = 0, x_3 - x_4 = 0$. The tropical line Γ_2 is contained in the linear space L defined by $x_2 - x_3 = 0, x_3 - x_4 = 0$. Consider the projection $\psi_{x_0,x_1,x_2} : \mathbb{P}^5 \to \mathbb{P}^2$ then $\mathrm{trop}(\phi^{-1}) = \phi^{-1}(\Gamma_2)$ is the tropical line with rays $\mathrm{pos}(1,0), \mathrm{pos}(0,1), \mathrm{pos}(-1,-1)$ and it is the tropicalization of the line $V(x_0+x_1+x_2)$. Applying ϕ we get that ℓ_2 is defined by $(x_0 + x_1 + x_2, x_2x_3 - x_4^2, x_3 - x_4)$.

Chapter 5

Computing tropical varieties

5.1 Introduction

The purpose of this package is to facilitate computations in tropical geometry using `Macaulay2` [GS]. At the moment the main tool for tropical geometry is the program `Gfan` [Jen] by Jensen. This computes the Gröbner fan of an ideal I and includes functions to compute only the subfan of the Gröbner fan given by the tropicalization of the variety $V(I)$. The polyhedral geometry program `Polymake` [GJ] also has some tropical functionality that is not implemented in `Gfan`.

The package `gfanInterface2` [HS], implemented in `Macaulay2`, allows the user to interface with `Gfan` while retaining the computational speed provided by `Macaulay2` for Gröbner basis computations. A drawback of this package is that it requires good knowledge of the functions and conventions of `Gfan`. The goal of the `Tropical` package is to provide a user friendly tool to do these computations in `Macaulay2` without requiring any knowledge of these conventions. The package includes different strategies for the same function depending on the input, and calls functions from `Gfan`, via `gfanInterface2`, and `Polymake`, as appropriate. Moreover the package implements some extra functionality not yet available in `Gfan`, such as computing multiplicities for tropical varieties of non-prime ideals and allowing the user to swap between the min and max conventions.

The package is available at

`http://homepages.warwick.ac.uk/staff/D.Maclagan/papers/TropicalPackage.html`

5.2 Use of the package

In this section we give explicit examples in order to give a short overview of the package. The computations are all over the field $\mathbb{Q}$ of rational numbers with trivial valuation, hence all tropical varieties are polyhedral fans.

The main function of the `Tropical` package is `tropicalVariety(I)`. This takes as input an ideal I in a polynomial ring $\mathbb{K}[x_1, \dots, x_n]$. The output is the tropicalization $\mathrm{trop}(V(I) \cap T^n)$ of $V(I) \subset \mathbb{A}^n$.

Example 5.2.1. Consider the algebraic variety $X = V(I) \subset \mathbb{A}^2 \cap T^2$, where $I = \langle x + y + 1 \rangle$. The tropicalization of this variety can be computed using the function `tropicalVariety(I)`. The package outputs this as a *tropical cycle*: a fan with a list of multiplicities corresponding to integer weights on the maximal cones. We extract information about the tropical cycle using associated functions. For example `rays` gives the generators of the rays as the columns of a matrix.

```
i1 : needsPackage "Tropical";
i2 : R=QQ[x,y];
i3 : I=ideal(x+y+1);
i4 : T=tropicalVariety I;
o4 = T
o4 : TropicalCycle
i5 : rays T
o5 = | -1 1 0 |
     | -1 0 1 |
                2        3
o5 : Matrix ZZ  <--- ZZ
```

```
i6 : linealitySpace T
o6 = 0
                3
o6 : Matrix ZZ  <--- 0
i7 : maxCones T
o7 = {{0}, {1}, {2}}
o7 : List
i8 : multiplicities T
o8 = {1, 1, 1}
o8 : List
```

The tropical variety $\mathrm{trop}\, V(I)$ is the standard tropical line in the plane: a 1–dimensional fan in $\mathbb{R}^2$ whose rays are $(-1,-1), (1,0)$, and $(0,1)$.

The function `tropicalVariety` uses one of two different algorithms depending on the input ideal. If the ideal is prime, the tropical variety is connected through codimension one ([BJS+07, Theorem 3.1]) and the `Gfan` commands `gfan_tropicalstartingcone` and `gfan_tropicaltraverse`, which implement the algorithm described in [BJS⁺07], are used. However if the ideal is not prime, this algorithm might fail. The package then calls the more computationally expensive command `gfan_tropicalbruteforce`, which computes the entire Gröbner fan. The multiplicities are then computed separately. The package does not require that the user knows these intricacies, but simply requires that they flag when the ideal is not prime.

`i9 : elapsedTime(`
`tropicalVariety I);`
`-- 0.088835 seconds elapsed`
	`i10 : elapsedTime(`
	`tropicalVariety(I,Prime=>false));`
	` -- 0.103651 seconds elapsed`

For most functions `Gfan` requires the input to be homogeneous. The `Tropical` package will accept non-homogeneous input, and do the pre- and post-processing to put it into a format acceptable for `Gfan`. Small additions such as this help decrease the prerequisite knowledge for the package.

Example 5.2.2. A tropical cycle is a fan with multiplicities attached to its maximal cones; it need not be the tropicalization of an algebraic variety. Therefore the package allows the user to create a tropical cycle manually by defining a fan via its maximal cones and attaching multiplicities to each of those cones. The following example shows how we can construct manually $\operatorname{trop} V(I)$ of Example 5.2.1.

`i11 : C1=posHull(matrix{{1},{0}});`	`o15 : List`
`i12 : C2=posHull(matrix{{0},{1}});`	`i16 : S=tropicalCycle(F,mult)`
`i13 : C3=posHull(matrix{{-1},{-1}});`	`o16 = S`
`i14 : F=fan({C1,C2,C3})`	`o16 : TropicalCycle`
`o14 = F`	`i17 : isBalanced S`
`o14 : Fan`	`o17 : true`
`i15 : mult={1,1,1}`	
`o15 = {1, 1, 1}`	

The `tropicalCycle` command does not check that the resulting weighted fan is balanced. To verify this we use the `isBalanced` command.

Example 5.2.3. Consider the tropical hypersurfaces $\operatorname{trop} V(f)$ and $\operatorname{trop} V(g)$ cut out by the polynomials $f = x + y + z$ and $g = x^2 + y^2 + z^2$. Their intersection cuts out a tropical prevariety. We would like to compute whether this prevariety is equal to the tropical variety $\operatorname{trop} V(I)$ where $I = \langle f, g \rangle$.

```
i18 : R=QQ[x,y,z];

i19 : f=x+y+z;

i20 : g=x^2+y^2+z^2;

i21 : l={f,g};

i22 : Tp=tropicalPrevariety l;

i23 : Tv=tropicalVariety ideal l;
```

```
i24 : isTropicalBasis l

o24 = false

i25 : dim Tp

o25 = 2

i26 : dim Tv

o26 = 1
```

The polynomials f, g are not a tropical basis for I and therefore the prevariety given by them is not equal to $\operatorname{trop} V(I)$. We can see from our computation that the prevariety has a two-dimensional cone, while $\operatorname{trop} V(I)$ is one-dimensional.

Example 5.2.4. For two curves $V(f)$ and $V(g)$ in $\mathbb{P}^2$, Bézout's Theorem states that $|V(f) \cap V(g)|$ equals $\deg(f) \cdot \deg(g)$ counting multiplicities. The tropical analogue of Bézout's Theorem states that the *stable intersection* of $\operatorname{trop} V(f)$ and $\operatorname{trop} V(g)$ is $\deg(f) \cdot \deg(g)$ points counting multiplicities. The following example shows how the package and the `stableIntersection` function can be used to verify examples of tropical Bézout's Theorem.

```
i27 : f=random(2,R);

i28 : g=random(1,R);

i29 : Tf=tropicalVariety ideal f;

i30 : Tg=tropicalVariety ideal g;

i31 : Tint=stableIntersection(Tf,Tg)

o31 = Tint

o31 : TropicalCycle

i32 : rays Tint
```

```
o32 = 0

                  3
o32 : Matrix ZZ  <--- 0

i33 : maxCones Tint

o33 = {{}}

o33 : List

i34 : multiplicities Tint

o34 = {2}

o34 : List
```

The above code considers the stable intersection of a tropical line and a plane quadric. The resulting tropical cycle is a single point, the origin, with multiplicity two, verifying the claim of tropical Bézout's theorem.

The function `stableIntersection` has two different strategies for computation depending on the software available to the user. If the user has a recent version of `Polymake` installed, the default strategy is to use `atint` [Ham14], a `Polymake` extension for tropical intersection theory by Simon Hampe. If this is not available, the package instead uses `Gfan` to compute the stable intersection.

5.3 Future Plans

We plan for the `Tropical` package to become the umbrella package for all tropical computations in `Macaulay2`. This will include implementing alternate strategies for some of the core commands as algorithms improve, before they are included into `Gfan` and `Polymake`.

In addition there are still functions available in `Gfan` and `Polymake` that are not yet available in the package. We particularly highlight the treatment of nontrivial valuations, which is available in `Gfan`, and the visualization of low-dimensional tropical varieties, which is available in `Polymake`.

Appendix

In this Appendix we provide numerical evidence of our computations in Chapter 3. Table 5.1 contains data on the non-prime maximal cones of $\mathrm{trop}(\mathcal{F}\ell_5)$. In Table 5.2 there is information on the polytopes obtained from maximal prime cones of $\mathrm{trop}(\mathcal{F}\ell_5)$. This includes the F-vectors, combinatiral equivalences among the polytopes, and between those and the string polytopes, resp. FFLV polytope, for ρ.

Algebraic and combinatorial invariants of $\mathrm{trop}(\mathcal{F}\ell_5)$

Below we collect in a table all the information about the non-prime initial ideals of $\mathcal{F}\ell_5$ up to symmetry.

Number of Orbits	#Generators
30	69
267	66
37	68
11	70
10	71
2	73

Table 5.1: Data for non-prime initial ideals of $\mathcal{F}\ell_5$.

The following table shows the F-vectors of the polytopes associated to maximal prime cones of $\mathrm{trop}(\mathcal{F}\ell_5)$ for one representative in each orbit. The last column contains information on the existence of a combinatorial equivalence between these polytopes and the string polytopes resp. FFLV polytope for ρ. The initial ideals are all *Cohen-Macaulay*.

Orbit	F-vector	Combinatorial equivalences
0	475 2956 8417 14241 15690 11643 5820 1899 374 37	

1	456 2799 7843 13023 14038 10159 4938 1565 301 30	
2	425 2573 7108 11626 12333 8779 4201 1316 253 26	
3	393 2313 6200 9833 10125 7021 3297 1027 201 22	
4	433 2621 7230 11796 12473 8847 4219 1318 253 26	
5	435 2630 7246 11810 12479 8848 4219 1318 253 26	
6	425 2553 6988 11317 11888 8388 3987 1245 240 25	
7	450 2751 7677 12699 13648 9863 4800 1529 297 30	
8	435 2630 7246 11810 12479 8848 4219 1318 253 26	
9	419 2522 6922 11243 11842 8373 3985 1245 240 25	
10	453 2785 7817 12999 14027 10157 4938 1565 301 30	
11	463 2885 8237 13987 15474 11532 5788 1895 374 37	
12	463 2852 8020 13365 14459 10501 5121 1627 313 31	
13	457 2840 8078 13638 14954 10996 5413 1726 330 32	
14	454 2819 8016 13540 14870 10968 5427 1744 337 33	
15	445 2748 7770 13050 14254 10464 5161 1658 322 32	
16	441 2681 7438 12228 13056 9369 4525 1430 276 28	
17	440 2704 7602 12684 13752 10014 4897 1560 301 30	
18	471 2923 8298 13995 15369 11369 5667 1845 363 36	
19	464 2883 8200 13861 15258 11313 5651 1843 363 36	
20	467 2911 8309 14097 15574 11586 5804 1897 374 37	
21	461 2876 8225 13993 15509 11575 5814 1903 375 37	
22	397 2363 6416 10313 10755 7536 3561 1109 215 23	
23	437 2669 7447 12319 13236 9556 4642 1475 286 29	
24	425 2553 6988 11317 11888 8388 3987 1245 240 25	
25	415 2498 6861 11158 11772 8339 3976 1244 240 25	
26	470 2942 8436 14377 15944 11889 5955 1939 379 37	
27	460 2856 8109 13656 14929 10944 5374 1712 328 32	
28	449 2741 7634 12594 13487 9702 4695 1486 287 29	
29	427 2592 7181 11778 12523 8926 4270 1334 255 26	
30	425 2573 7108 11626 12333 8779 4201 1316 253 26	FFLV
31	443 2708 7557 12495 13411 9667 4686 1485 287 29	
32	397 2363 6416 10313 10755 7536 3561 1109 215 23	S22
33	425 2553 6988 11317 11888 8388 3987 1245 240 25	
34	419 2522 6922 11243 11842 8373 3985 1245 240 25	
35	405 2407 6518 10442 10851 7578 3571 1110 215 23	

36	401 2387 6477 10398 10825 7570 3570 1110 215 23	
37	368 2154 5755 9111 9373 6497 3052 953 188 21	S21
38	379 2214 5892 9280 9494 6547 3063 954 188 21	S27, S28
39	393 2313 6200 9833 10125 7021 3297 1027 201 22	
40	358 2069 5453 8516 8653 5941 2778 870 174 20	S1, S18, S26, S29
41	459 2851 8111 13720 15118 11223 5614 1834 362 36	
42	467 2913 8322 14133 15629 11636 5831 1905 375 37	
43	423 2562 7083 11596 12313 8772 4200 1316 253 26	
44	425 2573 7108 11626 12333 8779 4201 1316 253 26	S24
45	397 2363 6416 10313 10755 7536 3561 1109 215 23	S23
46	461 2876 8225 13993 15509 11575 5814 1903 375 37	
47	400 2366 6377 10175 10546 7363 3480 1089 213 23	
48	393 2313 6200 9833 10125 7021 3297 1027 201 22	
49	393 2313 6200 9833 10125 7021 3297 1027 201 22	
50	379 2214 5892 9280 9494 6547 3063 954 188 21	S2, S19
51	426 2599 7257 12034 12981 9420 4602 1470 286 29	
52	428 2594 7176 11761 12514 8947 4307 1359 263 27	
53	419 2522 6922 11243 11842 8373 3985 1245 240 25	
54	466 2917 8371 14288 15879 11870 5960 1944 380 37	
55	443 2729 7692 12867 13982 10197 4987 1585 304 30	
56	453 2787 7826 13011 14021 10122 4895 1539 293 29	
57	469 2926 8358 14188 15679 11663 5839 1906 375 37	
58	458 2825 7958 13286 14398 10472 5113 1626 313 31	
59	472 2949 8435 14335 15854 11796 5902 1923 377 37	
60	440 2704 7602 12684 13752 10014 4897 1560 301 30	
61	472 2967 8561 14720 16525 12526 6410 2144 432 43	
62	457 2842 8099 13726 15153 11266 5640 1842 363 36	
63	465 2902 8296 14096 15588 11594 5795 1884 368 36	
64	459 2851 8111 13720 15118 11223 5614 1834 362 36	
65	428 2608 7269 12028 12946 9377 4576 1462 285 29	
66	441 2681 7438 12228 13056 9369 4525 1430 276 28	
67	418 2510 6876 11157 11753 8321 3969 1243 240 25	
68	406 2442 6713 10943 11587 8245 3950 1241 240 25	
69	373 2199 5926 9474 9849 6897 3267 1024 201 22	
70	427 2586 7144 11681 12383 8806 4209 1317 253 26	

71	451 2781 7840 13111 14243 10390 5089 1623 313 31	
72	440 2704 7602 12684 13752 10014 4897 1560 301 30	
73	406 2442 6713 10943 11587 8245 3950 1241 240 25	
74	448 2764 7800 13061 14208 10377 5087 1623 313 31	
75	462 2873 8181 13846 15258 11321 5656 1844 363 36	
76	457 2842 8099 13726 15153 11266 5640 1842 363 36	
77	469 2927 8364 14203 15699 11678 5845 1907 375 37	
78	454 2802 7903 13216 14348 10453 5110 1626 313 31	
79	451 2787 7879 13221 14419 10565 5200 1667 323 32	
80	441 2705 7584 12611 13622 9885 4823 1537 298 30	
81	454 2803 7914 13263 14455 10598 5231 1687 330 33	
82	441 2697 7532 12465 13391 9660 4685 1485 287 29	
83	445 2721 7593 12550 13461 9694 4694 1486 287 29	
84	441 2697 7532 12465 13391 9660 4685 1485 287 29	
85	445 2725 7617 12611 13546 9764 4728 1495 288 29	
86	397 2363 6416 10313 10755 7536 3561 1109 215 23	
87	368 2154 5755 9111 9373 6497 3052 953 188 21	S5, S31
88	452 2801 7946 13385 14654 10771 5309 1699 327 32	
89	430 2624 7318 12097 12974 9329 4497 1411 269 27	
90	456 2834 8071 13670 15083 11210 5612 1834 362 36	
91	432 2633 7332 12104 12975 9341 4521 1430 276 28	
92	467 2919 8359 14230 15769 11756 5892 1922 377 37	
93	456 2834 8071 13670 15083 11210 5612 1834 362 36	
94	426 2597 7244 11998 12926 9370 4575 1462 285 29	
95	440 2708 7630 12769 13898 10169 5001 1603 311 31	
96	432 2633 7332 12104 12975 9341 4521 1430 276 28	
97	412 2479 6810 11083 11707 8306 3967 1243 240 25	
98	415 2511 6945 11391 12133 8679 4174 1313 253 26	
99	458 2845 8092 13676 15042 11132 5543 1800 353 35	
100	437 2669 7447 12319 13236 9556 4642 1475 286 29	
101	441 2703 7569 12562 13531 9780 4746 1502 289 29	
102	427 2586 7144 11681 12383 8806 4209 1317 253 26	
103	419 2522 6922 11243 11842 8373 3985 1245 240 25	
104	437 2669 7447 12319 13236 9556 4642 1475 286 29	
105	411 2470 6776 11012 11617 8235 3933 1234 239 25	

106	413 2483 6808 11043 11606 8177 3871 1201 230 24	
107	425 2553 6988 11317 11888 8388 3987 1245 240 25	
108	405 2407 6518 10442 10851 7578 3571 1110 215 23	
109	405 2427 6638 10751 11296 7969 3785 1181 228 24	S30
110	465 2904 8312 14152 15700 11734 5907 1940 384 38	
111	464 2902 8323 14204 15795 11828 5960 1956 386 38	
112	438 2690 7559 12608 13667 9952 4868 1552 300 30	
113	445 2725 7617 12611 13546 9764 4728 1495 288 29	
114	437 2669 7447 12319 13236 9556 4642 1475 286 29	
115	411 2470 6776 11012 11617 8235 3933 1234 239 25	
116	424 2574 7139 11737 12529 8983 4332 1367 264 27	
117	419 2522 6922 11243 11842 8373 3985 1245 240 25	
118	401 2387 6477 10398 10825 7570 3570 1110 215 23	
119	405 2427 6638 10751 11296 7969 3785 1181 228 24	S6
120	464 2893 8261 14019 15483 11503 5746 1869 366 36	
121	454 2806 7928 13283 14448 10543 5159 1641 315 31	
122	451 2794 7928 13370 14676 10840 5387 1746 342 34	
123	444 2736 7715 12915 14053 10273 5044 1613 312 31	
124	466 2909 8318 14138 15644 11650 5837 1906 375 37	
125	456 2815 7939 13271 14398 10480 5118 1627 313 31	
126	423 2561 7078 11586 12303 8767 4199 1316 253 26	
127	429 2580 7064 11429 11972 8402 3959 1221 232 24	
128	431 2626 7309 12058 12915 9290 4494 1422 275 28	
129	428 2602 7224 11883 12684 9087 4375 1377 265 27	
130	443 2727 7679 12831 13927 10147 4960 1577 303 30	
131	432 2637 7354 12152 13024 9356 4505 1412 269 27	
132	451 2793 7920 13342 14620 10770 5331 1718 334 33	
133	434 2632 7273 11879 12557 8883 4210 1301 246 25	
134	452 2781 7813 13004 14042 10171 4944 1566 301 30	
135	453 2808 7969 13433 14725 10847 5366 1727 335 33	
136	451 2794 7928 13370 14676 10840 5387 1746 342 34	
137	433 2646 7390 12236 13150 9482 4589 1448 278 28	
138	442 2715 7629 12727 13808 10076 4948 1587 309 31	
139	432 2633 7332 12104 12975 9341 4521 1430 276 28	
140	423 2564 7096 11632 12368 8822 4227 1324 254 26	

141	413 2483 6808 11043 11606 8177 3871 1201 230 24	
142	427 2594 7196 11827 12614 9031 4347 1369 264 27	
143	431 2622 7281 11973 12769 9135 4390 1379 265 27	
144	431 2626 7309 12058 12915 9290 4494 1422 275 28	
145	410 2459 6725 10881 11411 8029 3802 1183 228 24	
146	428 2594 7176 11761 12514 8947 4307 1359 263 27	
147	419 2522 6922 11243 11842 8373 3985 1245 240 25	
148	451 2781 7840 13111 14243 10390 5089 1623 313 31	
149	464 2900 8310 14168 15740 11778 5933 1948 385 38	
150	446 2750 7757 12985 14123 10315 5058 1615 312 31	
151	420 2541 7021 11496 12218 8719 4184 1314 253 26	
152	441 2705 7584 12611 13622 9885 4823 1537 298 30	
153	425 2575 7119 11651 12363 8799 4208 1317 253 26	
154	448 2764 7801 13067 14223 10397 5102 1629 314 31	
155	444 2737 7724 12949 14124 10363 5115 1647 321 32	
156	452 2772 7753 12830 13755 9876 4750 1486 282 28	
157	442 2706 7565 12529 13460 9696 4684 1473 281 28	
158	441 2708 7602 12655 13676 9915 4821 1525 292 29	
159	427 2596 7207 11850 12633 9026 4324 1350 257 26	
160	452 2781 7813 13004 14042 10171 4944 1566 301 30	
161	427 2586 7144 11681 12383 8806 4209 1317 253 26	
162	400 2382 6467 10388 10820 7569 3570 1110 215 23	
163	448 2764 7800 13061 14208 10377 5087 1623 313 31	
164	470 2943 8444 14405 16000 11959 6011 1967 387 38	
165	460 2857 8117 13684 14985 11014 5430 1740 336 33	
166	418 2530 6996 11466 12198 8712 4183 1314 253 26	
167	434 2640 7325 12025 12788 9108 4348 1353 257 26	
168	425 2577 7132 11687 12418 8849 4235 1325 254 26	
169	425 2581 7160 11772 12564 9004 4339 1368 264 27	
170	430 2614 7255 11928 12724 9109 4382 1378 265 27	
171	422 2557 7075 11597 12333 8801 4220 1323 254 26	
172	411 2470 6772 10988 11556 8150 3863 1200 230 24	S7
173	427 2586 7144 11681 12383 8806 4209 1317 253 26	
174	400 2382 6467 10388 10820 7569 3570 1110 215 23	
175	464 2898 8295 14119 15649 11673 5856 1913 376 37	

176	442 2718 7644 12754 13822 10056 4911 1562 301 30	
177	440 2698 7563 12576 13587 9864 4816 1536 298 30	
178	423 2562 7083 11596 12313 8772 4200 1316 253 26	
179	452 2781 7813 13004 14042 10171 4944 1566 301 30	

Table 5.2: Orbits of maximal prime cones for $\mathcal{F}\ell_5$, the F-vectors of the corresponding polytopes, and combinatorially equivalent string polytopes resp. FFLV polytope.

Computations for $\mathrm{F}_1(\operatorname{trop} X)$ of Example 4.3.4

The tropical Fano scheme $\mathrm{F}_1(\operatorname{trop} L)$ is the tropical prevariety obtained by intersecting the following polynomials:

$$
\begin{aligned}
&p_{34}p_{25} - p_{24}p_{35} + p_{23}p_{45}, p_{34}p_{15} - p_{14}p_{35} + p_{13}p_{45}, p_{24}p_{15} - p_{14}p_{25} + p_{12}p_{45},\\
&p_{23}p_{15} - p_{13}p_{25} + p_{12}p_{35}, p_{34}p_{05} - p_{04}p_{35} + p_{03}p_{45}, p_{24}p_{05} - p_{04}p_{25} + p_{02}p_{45},\\
&p_{14}p_{05} - p_{04}p_{15} + p_{01}p_{45}, p_{23}p_{05} - p_{03}p_{25} + p_{02}p_{35}, p_{13}p_{05} - p_{03}p_{15} + p_{01}p_{35},\\
&p_{12}p_{05} - p_{02}p_{15} + p_{01}p_{25}, p_{23}p_{14} - p_{13}p_{24} + p_{12}p_{34}, p_{23}p_{04} - p_{03}p_{24} + p_{02}p_{34},\\
&p_{13}p_{04} - p_{03}p_{14} + p_{01}p_{34}, p_{12}p_{04} - p_{02}p_{14} + p_{01}p_{24}, p_{12}p_{03} - p_{02}p_{13} + p_{01}p_{23},\\
&98p_{25} - 74p_{35} - 58p_{45}, 98p_{15} + 80p_{35} - 128p_{45}, 98p_{05} - 52p_{35} - 54p_{45},\\
&74p_{15} + 80p_{25} - 144p_{45}, -58p_{15} + 128p_{25} - 144p_{35},\\
&98p_{12} - 74p_{13} - 80p_{23} - 58p_{14} + 128p_{24} - 144p_{34},\\
&74p_{05} - 52p_{25} - 10p_{45}, -58p_{05} + 54p_{25} - 10p_{35},\\
&98p_{02} - 74p_{03} + 52p_{23} - 58p_{04} + 54p_{24} - 10p_{34},\\
&-80p_{05} - 52p_{15} + 112p_{45}, -128p_{05} + 54p_{15} + 112p_{35}, -144p_{05} + 10p_{15} + 112p_{25},\\
&98p_{01} + 80p_{03} + 52p_{13} - 128p_{04} + 54p_{14} + 112p_{34},\\
&74p_{01} + 80p_{02} + 52p_{12} - 144p_{04} + 10p_{14} + 112p_{24},\\
&-58p_{01} + 128p_{02} - 54p_{12} - 144p_{03} + 10p_{13} + 112p_{23},\\
&98p_{24} - 74p_{34} + 153p_{45}, 98p_{23} + 58p_{34} + 153p_{35},\\
&74p_{23} + 58p_{24} + 153p_{25}, -98p_{12} + 74p_{13} + 58p_{14} + 153p_{15},\\
&-98p_{02} + 74p_{03} + 58p_{04} + 153p_{05}, 98p_{14} + 80p_{34} + 461p_{45}, 98p_{13} + 128p_{34} + 461p_{35},\\
&98p_{12} - 80p_{23} + 128p_{24} + 461p_{25}, -80p_{13} + 128p_{14} + 461p_{15},\\
&-98p_{01} - 80p_{03} + 128p_{04} + 461p_{05}, 98p_{04} - 52p_{34} - 13p_{45}, 98p_{03} + 54p_{34} - 13p_{35},
\end{aligned}
$$

$98p_{02} + 52p_{23} + 54p_{24} - 13p_{25}, 98p_{01} + 52p_{13} + 54p_{14} - 13p_{15},$

$52p_{03} + 54p_{04} - 13p_{05}, 74p_{14} + 80p_{24} + 473p_{45}, 98p_{12} - 58p_{14} + 128p_{24} + 473p_{35},$

$74p_{12} + 144p_{24} + 473p_{25}, -80p_{12} + 144p_{14} + 473p_{15},$

$-74p_{01} - 80p_{02} + 144p_{04} + 473p_{05}, 153p_{14} - 461p_{24} + 473p_{34}, 74p_{04} - 52p_{24} - 91p_{45},$

$98p_{02} - 58p_{04} + 54p_{24} - 91p_{35}, 74p_{02} + 10p_{24} - 91p_{25}, 74p_{01} + 52p_{12} + 10p_{14} - 91p_{15},$

$52p_{02} + 10p_{04} - 91p_{05},$

$153p_{04} + 13p_{24} - 91p_{34}, -80p_{04} - 52p_{14} - 234p_{45}, 98p_{01} - 128p_{04} + 54p_{14} - 234p_{35},$

$74p_{01} - 144p_{04} + 10p_{14} - 234p_{25}, -80p_{01} - 112p_{14} - 234p_{15},$

$52p_{01} - 112p_{04} - 234p_{05}, 461p_{04} + 13p_{14} - 234p_{34}, 473p_{04} + 91p_{14} - 234p_{24},$

$153p_{01} - 461p_{02} - 13p_{12} + 473p_{03} + 91p_{13} - 234p_{23},$

$-98p_{12} + 74p_{13} + 80p_{23} - 73p_{45}, -58p_{13} + 128p_{23} - 73p_{35}, -58p_{12} + 144p_{23} - 73p_{25},$

$-128p_{12} + 144p_{13} - 73p_{15},$

$58p_{01} - 128p_{02} + 144p_{03} - 73p_{05}, 153p_{13} - 461p_{23} - 73p_{34},$

$153p_{12} - 473p_{23} - 73p_{24}, 461p_{12} - 473p_{13} - 73p_{14},$

$-153p_{01} + 461p_{02} - 473p_{03} - 73p_{04}, -98p_{02} + 74p_{03} - 52p_{23} + 92p_{45},$

$-58p_{03} + 54p_{23} + 92p_{35}, -58p_{02} + 10p_{23} + 92p_{25}, -58p_{01} - 54p_{12} + 10p_{13} + 92p_{15},$

$-54p_{02} + 10p_{03} + 92p_{05}, 153p_{03} + 13p_{23} + 92p_{34},$

$153p_{02} + 91p_{23} + 92p_{24}, 153p_{01} - 13p_{12} + 91p_{13} + 92p_{14},$

$-13p_{02} + 91p_{03} + 92p_{04}, -98p_{01} - 80p_{03} - 52p_{13} + 271p_{45},$

$-128p_{03} + 54p_{13} + 271p_{35}, -58p_{01} - 144p_{03} + 10p_{13} + 271p_{25},$

$-128p_{01} - 112p_{13} + 271p_{15}, -54p_{01} - 112p_{03} + 271p_{05}, 461p_{03} + 13p_{13} + 271p_{34},$

$153p_{01} + 473p_{03} + 91p_{13} + 271p_{24}, 461p_{01} + 234p_{13} + 271p_{14},$

$-13p_{01} + 234p_{03} + 271p_{04}, -73p_{03} - 92p_{13} + 271p_{23},$

$-74p_{01} - 80p_{02} - 52p_{12} + 241p_{45}, 58p_{01} - 128p_{02} + 54p_{12} + 241p_{35}, -144p_{02} + 10p_{12} + 241p_{25},$

$-144p_{01} - 112p_{12} + 241p_{15}, -10p_{01} - 112p_{02} + 241p_{05}, -153p_{01} + 461p_{02} + 13p_{12} + 241p_{34},$

$473p_{02} + 91p_{12} + 241p_{24},$

$473p_{01} + 234p_{12} + 241p_{14}, -91p_{01} + 234p_{02} + 241p_{04},$

$-73p_{02} - 92p_{12} + 241p_{23}, -73p_{01} - 271p_{12} + 241p_{13}, 92p_{01} - 271p_{02} + 241p_{03}.$

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